BYJU'S EXAM PREP

# Electrical Engineering

## Complete Syllabus

## FORMULA BOOK

BYJU'S EXAM PREP

# Electrical Engineering

## Signals & Systems

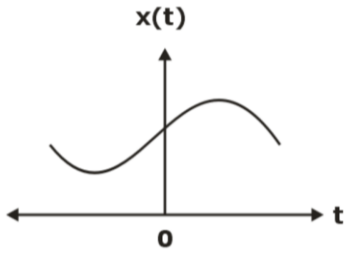
SHORT NOTES

# IMPORTANT FORMULAS TO REMEMBER

## CHAPTER 1: BASICS OF SIGNALS

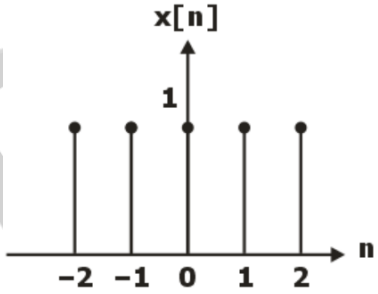
### 1. Continuous-time signal

A signal x(t) is continuous-time (CT) signal, if t is a continuous variable. A continuous time signal is defined continuously with respect to time.




### 2. Discrete-time signal

If t is a discrete variable, then it is a discrete-time (DT) signal. A discrete time signal is often identified as a sequence of numbers, denoted by x[n], where n is an integer.

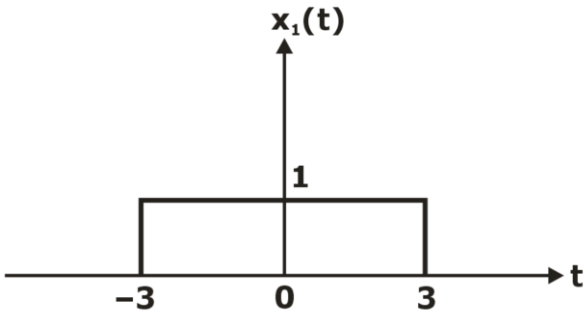


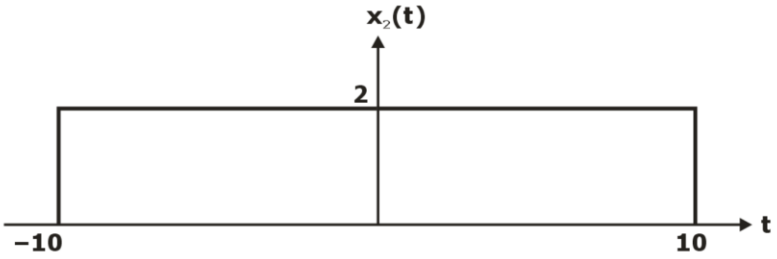

### 3. BASIC OPERATIONS ON CONTINUOUS TIME SIGNAL

#### 3.1. Addition/Subtraction of signals

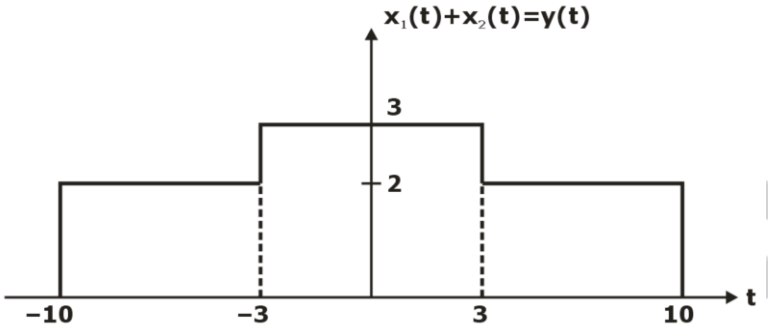
The sum of two continuous-time signals can be obtained by adding their values at every instant of time. Similarly, the subtraction of two continuous-time signals can be obtained by subtracting their values at every instant of time.

**Example:**






**(i) Addition**

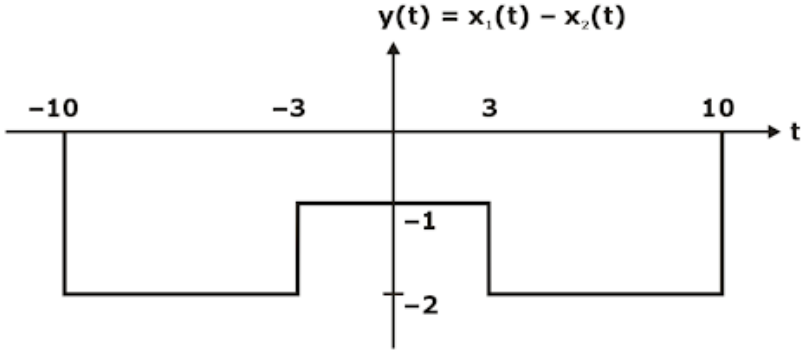



From −10 < t < −3, amplitude of $y(t) = x_1(t) + x_2(t) = 0 + 2 = 2$

From −3 < t < −3, amplitude of $y(t) = x_1(t) + x_2(t) = 1 + 2 = 3$

From 3 < t < 10 amplitude of $y(t) = x_1(t) + x_2(t) = 0 + 2 = 2$

**(ii) Subtraction**




From −10 < t < −3, amplitude of $y(t) = x_1(t) - x_2(t) = 0 - 2 = -2$

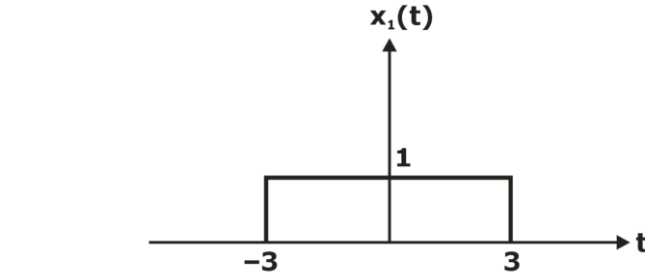
From −3 < t < −3, amplitude of $y(t) = x_1(t) - x_2(t) = 1 - 2 = -1$

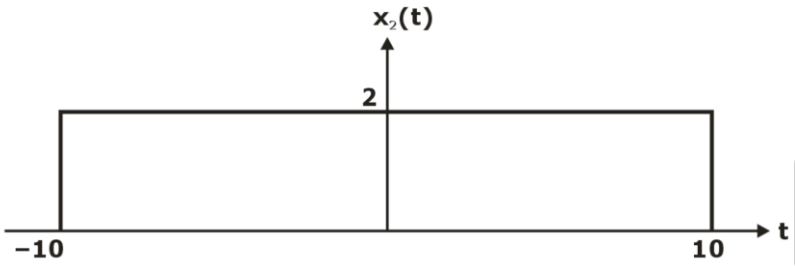
From 3 < t < 10, amplitude of $y(t) = x_1(t) - x_2(t) = 0 - 2 = -2$

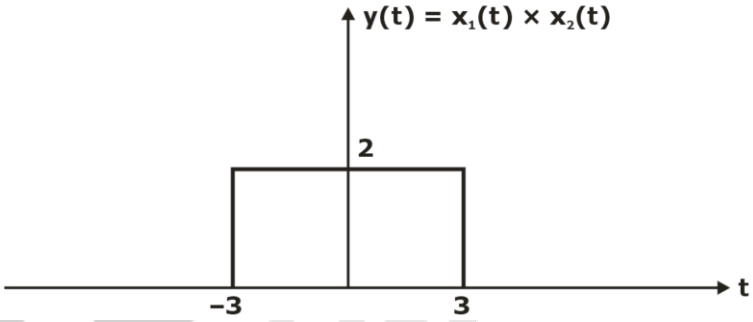
### 3.2. Multiplication of signals

The multiplication of two continuous signals can be obtained by multiplying their values at every instant.

**Example:**




From -10 < t < -3, amplitude of $y(t) = x_1(t) \times x_2(t) = 0 \times 2 = 0$

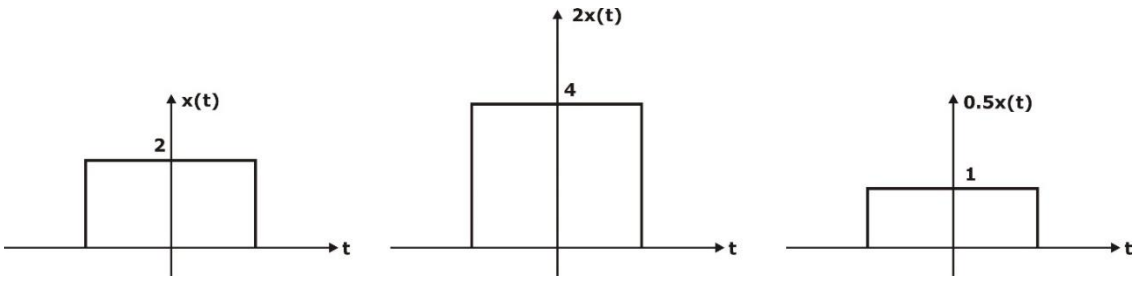
From -3 < t < 3, amplitude of $y(t) = x_1(t) \times x_2(t) = 1 \times 2 = 2$

From 3 < t < 10, amplitude of $y(t) = x_1(t) \times x_2(t) = 0 \times 2 = 0$

### 3.3. Amplitude scaling of signals

The amplitude of a signal can be changed by amplitude scaling. If a signal x(t) is multiplied by a factor A, it is expressed as A x(t) which means that, at every instant t, the amplitude of x(t) is multiplied by A.

**Example:**




**Note:**

Amplitude scaling signal A x(t) is identical in shape to the original signal x(t) but its amplitude is multiplied by A everywhere.

### 3.4. Transformation of signal

#### i. Time-Shifting

Signal $x(t - t_0)$ represents a time shifted version of $x(t)$ by $t_0$ seconds. If $t_0 > 0$, then the signal is delayed by $t_0$ seconds. If $t_0 < 0$, then $x(t + t_0)$ represents an advanced version of $x(t)$. The time shifting operation is shown in figure.

The waveform of $x(t - t_0)$ is identical to that of $x(t)$, except for a shift of $t_0$ time units towards the right-hand side.

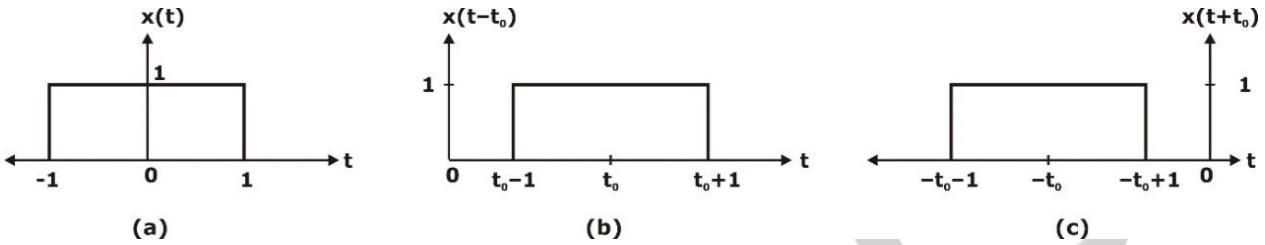



**Figure: Time shifting operation (a) Original signal x(t) (b) Time delayed version of x(t) (c) Time advanced version of x(t)**

#### ii. Time scaling

If the independent variable t is scaled by a parameter a, then x(at) is time scaled version of x(t). It is important to note that time scaling is performed on t-axis such that the values x(t) and x(at) at t = 0 are the name for both waveforms.

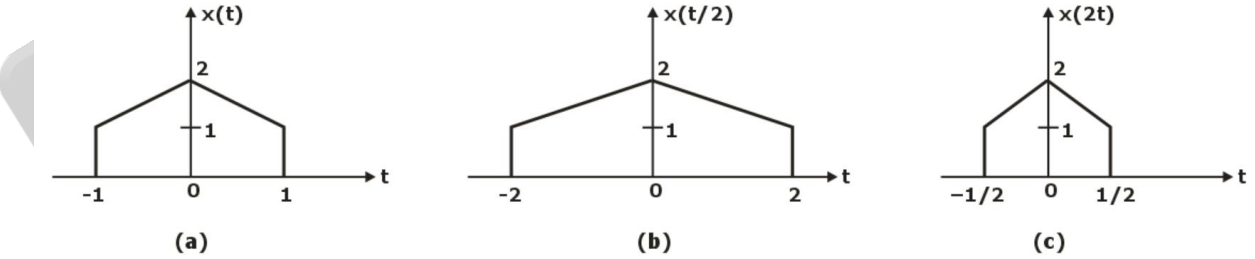
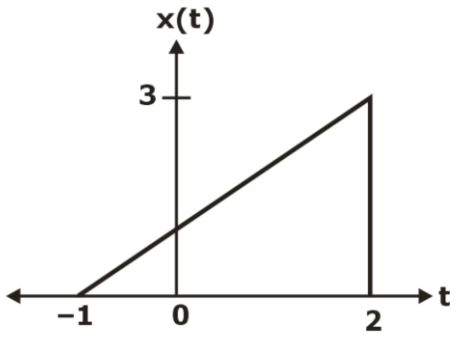


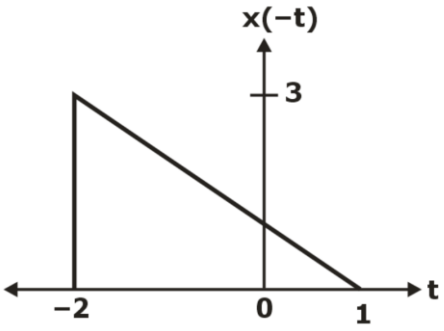

**Figure: Time scaling operations, (a) Original signal x(t), (b) Time expanded version, (c) Time compressed version of x(t)**

#### iii. Time-Reversal/Folding

The signal $x(-t)$ is called folded version of signal $x(t)$ and is obtained by taking reflection of $x(t)$ about vertical axis $t = 0$ as shown in figure.




(a) Signal x(t)




(b) Reflection of x(t)

## 4. MULTIPLE OPERATIONS ON CONTINEOUS-TIME SIGNALS

Consider a signal x(t) with multiple transformation given as

$x(t) \rightarrow A\,x(bt \pm t_0)$

where, a and b are assumed to be real numbers. The operations should be performed in the following order

**METHODOLOGY 1**

**Step 1:** First multiply signal by a constant A to obtained amplitude scaled version of x(t) that is A x(t).

**Step 2:** Shift the signal A x(t) to the left or to right by $t_0$ time units. This will produced shifted signal A x(t ± $t_0$).

**Step 3:** Scale the signal A x(t ± $t_0$) by b, the resulting signal represents Ax(bt ± $t_0$).

**Step 4:** If b is negative, reflect the scaled signal Ax(bt ± $t_0$) about the vertical axis.

The correct sequence for the above transformation is

$$x(t) \xrightarrow[\text{scaling}]{A\ \text{amplitude}} Ax(t) \xrightarrow[t\rightarrow t-t_0\ \text{shifting}]{\text{time}} x(t-t_0) \xrightarrow[t\rightarrow bt\ \text{scaling}]{\text{time}} Ax(bt-t_0)$$

If time scaling is done before time shifting it will produce incorrect results.

$$x(t) \xrightarrow[\text{scaling}]{A\ \text{amplitude}} Ax(t) \xrightarrow[t\rightarrow bt\ \text{scaling}]{\text{time}} Ax(bt) \xrightarrow[t\rightarrow t-t_0\ \text{shifting}]{\text{time}} Ax[b(t-t_0)] \neq Ax(bt-t_0)$$

For a different multiple transformation, different order of sequence is performed. Consider a signal x(t) with multiple transformation given as

$x(t) \rightarrow Ax\left(\frac{t-t_0}{a}\right)$

For this sequence the simplest sequence of operation is given as follows.

**METHODOLOGY 2**

**Step 1:** First multiply signal by a constant A to obtained amplitude scaled version of x(t) that is Ax(t).

**Step 2:** Scale the signal Ax(t) by 1/a, the resulting signal represents x(t/a).

**Step 3:** If a is negative, reflect the scaled signal x(t/a) about the vertical axis.

**Step 4:** Shift the scaled signal Ax(t/a) by $t_0$ units to the left or to right by $t_0$ time units. This will produced signal Ax[(t – $t_0$) /a].

The correct sequence for the above transformation is

$$x(t) \xrightarrow{A_{scaling}^{amplitude}} Ax(t) \xrightarrow{t\to t/a_{scaling}^{time}} Ax\left(\frac{t}{a}\right) \xrightarrow{t\to t-t_0{}_{shifting}^{time}} Ax\left(\frac{t-t_0}{a}\right)$$

If we change the order of sequence (time scaling is done after time shifting), then we would not get correct results.

$$x(t) \xrightarrow{A_{scaling}^{amplitude}} Ax(t) \xrightarrow{t\to t-t_0{}_{shifting}^{time}} Ax(t-t_0) \xrightarrow{t\to t/a_{scaling}^{time}} Ax\left(\frac{t}{a}-t_0\right) \neq Ax\left(\frac{t-t_0}{a}\right)$$

It must be noted that the operation of reflecting and time scaling is commutative, whereas the operation of shifting and reflecting or shifting and time scaling is not.

## 5. Some Important Signals

| Name | Continuous | Discrete |
|---|---|---|
| Unit Step function | $u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$ | $u[n]=\begin{cases}1, & n\geq 0\\ 0, & t<0\end{cases}$ |
| Ramp signal | $r[t]=\begin{cases}t, & t\geq 0\\ 0, & t<0\end{cases}$ | $r[n]=nu(n)=\begin{cases}n, & n\geq 0\\ 0, & n<0\end{cases}$ |
| Impulse function | $\delta(t)=0, t\neq 0$ | $\delta[n]=\begin{cases}1, & n=0\\ 0, & otherwise\end{cases}$ |
| Rectangular pulse function | $rect\left(\frac{t}{\tau}\right)=\begin{cases}1, & \lvert t\rvert\leq \tau/2\\ 0, & \lvert t\rvert>\tau/2\end{cases}$ | $rect\left[\frac{n}{2N}\right]=\begin{cases}1, & \lvert n\rvert\leq N\\ 0, & \lvert n\rvert>N\end{cases}$ |
| Triangular pulse | $tri\left(\frac{t}{\tau}\right)=\begin{cases}1-\left\lvert\frac{t}{\tau}\right\rvert, & t\leq\lvert\tau\rvert\\ 0, & t>\lvert\tau\rvert\end{cases}$ | $tri\left[\frac{n}{N}\right]=\begin{cases}1-\frac{\lvert n\rvert}{N}, & \lvert n\rvert\leq N\\ 0, & elsewhere\end{cases}$ |
| Signum signal | $sgn(t)=\begin{cases}1, & t>0\\ -1, & t<0\end{cases}$ | $sgn[n]=\begin{cases}1, & n>0\\ -1, & n<0\end{cases}$ |
| Sinusoidal signal | $x(t)=\sin(2\pi f_0 t+\theta)$ | $X[n]=\sin(2\pi f_0 n+\theta)$ |
| Sinc function | $sin(\omega_0 t)=\frac{sin(\pi\omega_0 t)}{\pi\omega_0 t}$ | $sin(\omega_0 n)=\frac{sin(\pi\omega_0 n)}{\pi\omega_0 n}$ |

## 6. Important Properties of Signals

| | | | |
|---|---|---|---|
| Signals in term of unit step and vice versa | r(t) = tu(t)<br>$u(t) = \frac{d}{dt}r(t)$<br>$\delta(t) = \frac{d}{dt}u(t)$<br>$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$<br>sgn = u(t) – u(–t)<br>sgn = 2u(t) – 1<br>$\prod\left(\frac{t}{\tau}\right) = u\left(t + \frac{t}{\tau}\right) - u\left(t - \frac{t}{\tau}\right)$ | Impulse properties | $\int_{-\infty}^{\infty} \delta(t)dt = 1$<br>$\delta(\alpha t) = \frac{1}{\lvert\alpha\rvert}\delta(t)$<br>$\delta(\alpha t + b) = \frac{1}{\lvert\alpha\rvert}\delta\left(t + \frac{b}{\alpha}\right)$<br>$\int_{-\infty}^{\infty} f(t)\delta(t - \lambda)dt = f(\lambda)$<br>$f(t)\delta(t - \lambda) = f(\lambda)\delta(t - \lambda)$ |
| Time period of linear combination of two signals | Sum of signals is periodic if $\frac{T_1}{T_2} = \frac{m}{n}$= rational number<br>The fundamental period of g(t) is given by $nT_1 = mT_2$ provided that the values of m and n are chosen such that the greastest common divisor (gcd) between m and n is 1 | odd and even & symmetry | $x_e(t) = x_e(-t)$<br>$x_0(t) = -x_0(-t)$<br>$x(t) = x_e(t) + x_0(t)$<br>$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$<br>$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$ |
| Combined operation | x(t) ⇒ Kx(t) + C<br>Scale by K then shift by C ….<br>x(t) ⇒ x(αt – β)<br>Shift by β : [x(t – β)] Then compress by a:[x (t – β) ⇒ x(αt – β)]<br>Or, compress by α: [x(t)⇒x(at)] then shift by $\frac{\beta}{\alpha}$:$x\left\{\alpha\left(t - \frac{\beta}{\alpha}\right)\right\}$= x (αt – β)}] | Derivative of impulse (doublet) | $\frac{d}{dt}\delta(t) = \delta'(t) = \begin{cases} undefined, & t = 0 \\ 0, & otherwise \end{cases}$<br>$\delta'(\alpha t) = \frac{1}{\alpha\lvert\alpha\rvert}\delta'(t)$<br>$\int_{-\infty}^{\infty} x(t)\delta'(t - \lambda)dt = -x'(\lambda)$<br>x(t)δ′(t) = x(0)δ′(t) – x′(0)δ(t) |
| Energy and power | Periodic signals have infinite energy hence power type signals. | | |

**Properties**

1. $x(t) \times \delta(t) = x(t)$
2. $x(t - \alpha) \times \delta(t - \beta) = x(t - \alpha - \beta)$
3. $\delta(t) \times \delta(t) = \delta(t)$
4. $[\delta(t) \times \delta(t) \times \delta(t) \times \text{-------}] = \delta(t)$
5. $\delta(t - \alpha) \times \delta(t - \beta) = \delta(t - \alpha - \beta)$

A. $u(t) \times u(t) = r(t)$

B. $u(t - \alpha) \times u(t - \beta) = r(t - \alpha - \beta)$

C. $u(t) \times u(t) = \rho(t) = \frac{t^2}{2}\, u(t)$

D. $r(t - \alpha) \times u(t - \beta) = \rho(t - \alpha - \beta) = \frac{(t - \alpha - \beta)^2}{2}\, u(t - \alpha - \beta)$

### 7. Gaussian function

The Gaussian function is defined by the expression

$$g_a(t) = e^{-at^2} \quad -\infty < t < \infty$$

The function is extremely useful in probability theory.

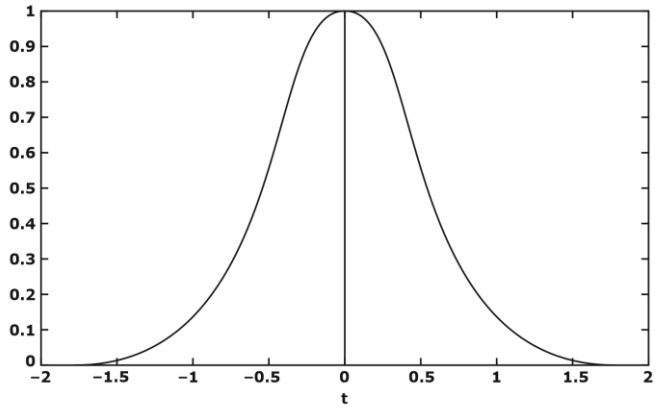



**Figure: Gaussian function**

### 8. Sinusoidal signal

A continuous-time sinusoidal signal is given by

$$x(t) = A \sin(\Omega t + \varphi)$$

where A is the amplitude, Ω is the frequency in radians per second and φ is the phase angle in radians.

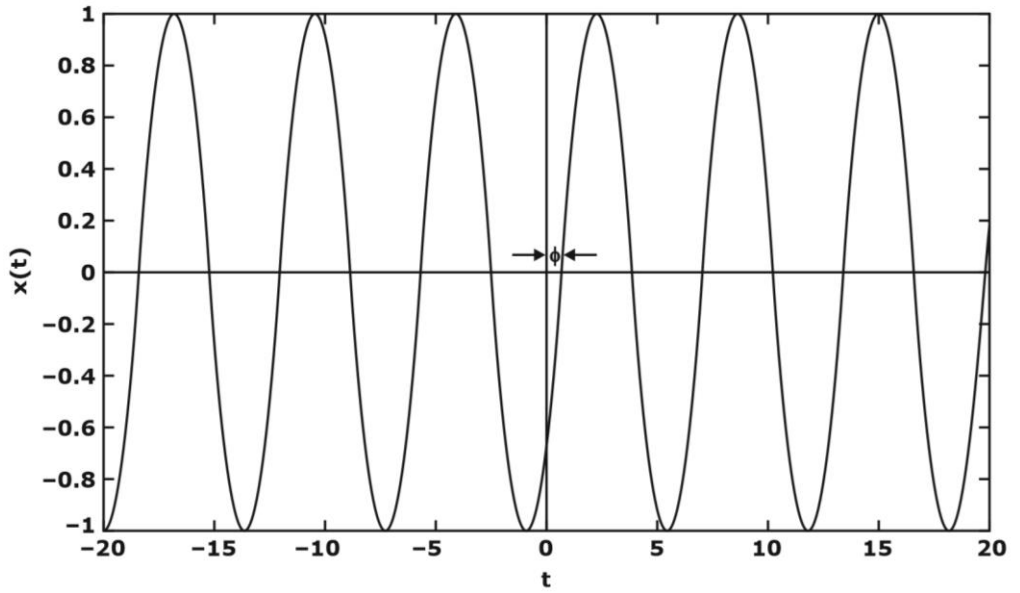



**Figure: Sinusoidal signal**

**9. Real exponential signal**

A real exponential signal is defined as

$$x(t) = Ae^{at}$$

where both A and a are real. Depending on the value of 'a' we get different signals.

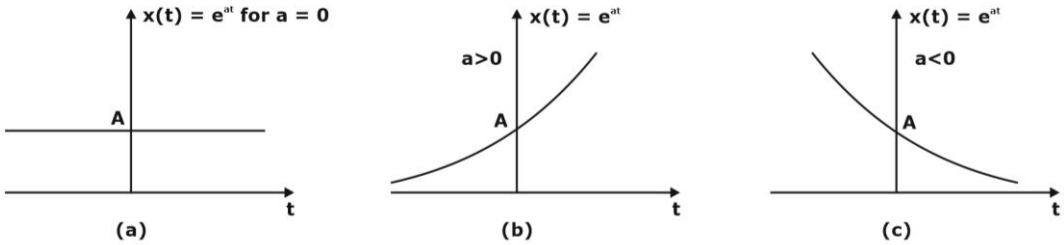



**Figure: (a) A dc signal (b) exponentially growing signal (c) exponentially decaying signal.**

**10. Complex exponential signal**

The most general form of complex exponential is given by

$x(t) = e^{st}$

Where, s is a complex variable defined as

$s = \sigma + j\Omega$

Depending on the values of σ and Ω, we get different signals.

**A.** If σ = 0 and Ω = 0 then x(t) = 1; that is the signal x(t) is a pure DC signal.

**B.** If Ω = 0, then s = σ and $x(t) = e^{\sigma t}$, which decays exponentially for σ < 0 and grows exponentially for σ > 0.

**C.** If σ = 0 then s = ±jΩ gives $x(t) = e^{j\Omega t}$ a sinusoidal signal with φ = 0.

**D.** If $\sigma < 0$ with finite $\Omega$, then x(t) is a exponentially decaying sinusoidal signal.

**E.** If $\sigma > 0$ with finite $\Omega$, then x(t) is a exponentially growing sinusoidal signal.

## 11. BASIC OPERATION ON DISCRETE TIME SIGNAL:

### 11.1. Addition of discrete-time signals

Addition of discrete time sequence is done by adding the signals at every instant of time [n].

### 11.2. Multiplication of discrete time signal

The multiplication of two discrete time signals $x_1[n]$ and $x_2[n]$ is obtained by multiplying the signal values at each instant of time n.

### 11.3. Amplitude scaling of discrete time Signals

Amplitude scaling is obtained by multiplying the signal x[n] with a constant A at each instant of time n. The amplitude-scaled is represented as A x[n].

### 11.4. Time-Scaling of discrete time Signals

Consider a discrete-time signal x[n], if the independent variable n is scaled by a factor of n then x[an] is the time-scaled version of x[n]. There are two types of time scaling

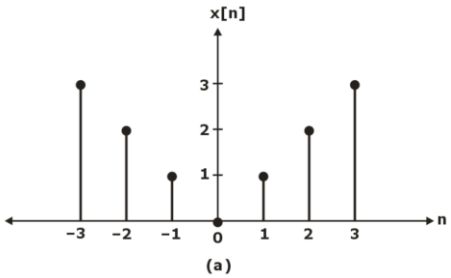
**Note:** Time-scaling of discrete time signals is different from continuous time signals, since discrete time signals are defined only for integer values of time variable n.

**Time Compression: Decimation or Down-sampling**

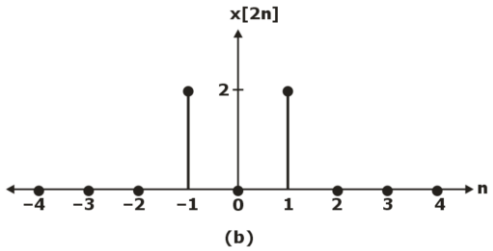
Compression of discrete time signals is also referred to as decimation. If a sequence x[n] is compressed by a factor a, some data samples of x[n] are lost. For example, if we compress x[n] by a factor of 2, the compressed signal y[n] = x[2n] contains only the alternate samples x[0], x[2], x[4] and so on. This operation losses data, and that is why time compression is called decimation or down-sampling.

**Time Expansion: Interpolation or Up-sampling**

In the discrete time domain, expansion is also referred to as interpolation. Let x[n] is expanded by a factor of 2 and the expanded signal is given as y[n] = x[n/2]. It is known that x[n] is defined only for integer value of n and zero for all non-integer values of n. Therefore, y[n] contains samples y[0] = x[0], y[2] = x[1], y[4] = x[2] and so on. The odd numbered samples y[1], y[3], y[5] all are zero.



(a)



(b)

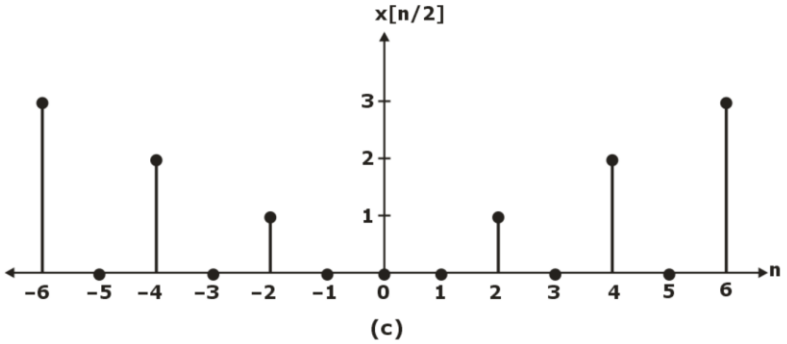



**Figure: Time scaling of DT signal, (a) Original DT sequence x[n], (b) Compressed (decimated) version of x[n], (c) Expanded (interpolated) version of x[n]**

### 11.5. Time-Shifting of discrete time Signals

The steps to obtain a time shifted signal from the original signal is given below.

(i) If x[n] is given, then $x[n + n_0]$ is plotted by shifting x[n] to the left by $n_0$.

(ii) If x[n] is given, then $x[n - n_0]$ is plotted by shifting x[n] to the right by $n_0$.

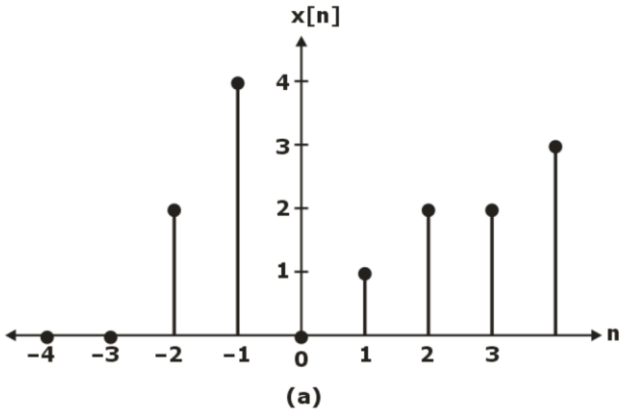
(iii) If x[−n] is given, then $x[- n - n_0]$ is plotted by shifting x[− n] to the left by $n_0$.

(iv) If x[− n] is given, then $x[- n + n_0]$ is plotted by shifting x[− n] to the right by $n_0$.

**Note:** The waveform of $x[n + n_0]$ is identical to that of x[n] except for a shift of $n_0$ time units towards the left-hand side.

### 11.6. Time-Reversal (folding) of discrete time signals

The folding operation produces a signal x[−n] which is the mirror image of x[n] about the vertical axis.



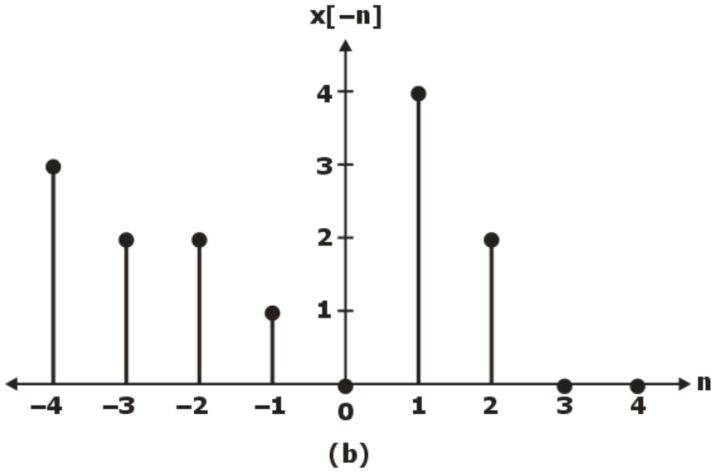



**Figure: Time folding of DT Signal, (a) Original DT Sequence x[n], (b) Folded Version of x[n].**

## 12. MULTIPLE OPERATIONS ON DISCRETE TIME SIGNALS

Consider a discrete time signal x[n] with multiple transformation given as

$x[n] \rightarrow A\, x[bn \pm n_0]$

Where a and b are assumed to be real numbers. The operations should be performed in the following order.

**Methodology**

**Step 1:** First multiply signal by a constant A to obtained amplitude scaled version of x[n] that is A x[n].

**Step 2:** Shift the signal A x[x] to the left or to right by $n_0$ time units. This will be produced shifted signal A x[n ± $n_0$].

**Step 3:** Scale the signal A x[n ± $n_0$] by b, the resulting signal represents A x[bn ± $n_0$].

**Step 4:** If b is negative, reflect the scaled signal A x[bn ± $n_0$] about the vertical axis.

The correct sequence for the above transformation is

$$x[n] \xrightarrow[\text{scaling}]{\text{A amplitude}} Ax[n] \xrightarrow[\text{shifting}]{\text{n}\rightarrow\text{n}-\text{n}_0\ \text{time}} Ax[n-n_0] \xrightarrow[\text{scaling}]{\text{n}\rightarrow\text{b}\ \text{time}} Ax[bn-n_0]$$

If time scaling is done before time shifting it will produce incorrect results.

$$x[n] \xrightarrow[\text{scaling}]{\text{A amplitude}} Ax[n] \xrightarrow[\text{shifting}]{\text{n}\rightarrow\text{bn}\ \text{time}} Ax[bn] \xrightarrow[\text{scaling}]{\text{n}\rightarrow\text{n}-\text{n}_0\ \text{time}} Ax[b(n-n_0)] \neq Ax[bn-n_0]$$

## 13. BASIC DISCRETE TIME SIGNALS

### 13.1. Discrete Impulse Function

The unit-impulse function in discrete time is defined as

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

$$\delta[n] = \{\ldots 0,0,0,\underset{\uparrow}{1},0,0,0\ldots\}$$

So, δ [n] is referred as the unit sample occurring at n = 0.

Similarly, for the shifted function δ [n – k] the unit sample occurring at n = k

That is,

$$\delta[n-k] = \begin{cases} 1, & n=k \\ 0, & n\neq k \end{cases}$$

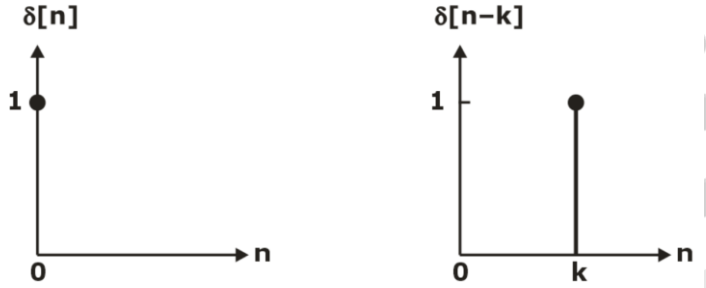



**Figure: (a) DT Unit Impulse Function (b) DT Shifted Unit Impulse Function**

**Properties**

Following are some of the important properties of unit impulse function.

**(i) Product property**

$$x[n]\ \delta[n - n_0] = x[n_0]\ \delta[n - n_0]$$

**(ii) Shifting property**

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$$

**(iii) Scaling Property**

The discrete-time unit impulse does not have a property corresponding to the scaling property of continuous-time unit impulse. Therefore, δ [n] = δ [an] for any nonzero integer value of n.

### 13.2. Discrete Unit Step Function

The unit-step sequence shown in figure 29 (a) is defined as,

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Or, $u[n] = \{\ldots 0,0,0,\underset{\uparrow}{1},1,1,1\ldots\}$

Similarly, the shifted unit-step sequence is defined as follows

$$u[n-k] = \begin{cases} 1, & n-k \geq 0 \ or\ n \geq k \\ 0, & n-k < 0 \ or\ n < k \end{cases}$$

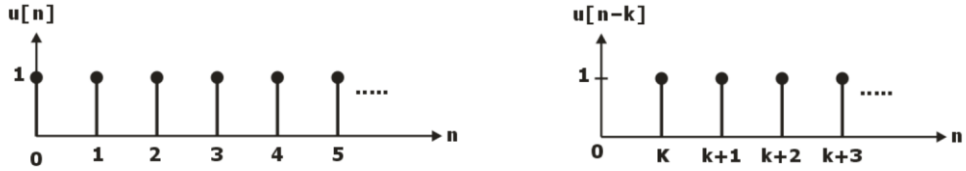



**Figure: (a) DT Unit Impulse Function,(b) Shifted DT Unit Impulse Function**

**Properties**

Following are some of the important properties of unit step function.

(i) δ[n] = u[n] – u[n – 1]

(ii) $\sum_{k=-\infty}^{\infty} \delta[k] = u[n]$

(iii) $\sum_{k=-\infty}^{\infty} \delta[n-k] = u[n]$

## 13.3. Discrete Unit-ramp Function

The unit ramp sequence is defined as

$$r[n] = \begin{cases} n \ for \ n \geq 0 \\ 0 \ for \ n < 0 \end{cases}$$

Or, $r[n] = nu[n] = \{0, 1,2,3,4,5,\dots\}$

The graphical representation of r(n) is shown in figure 30.

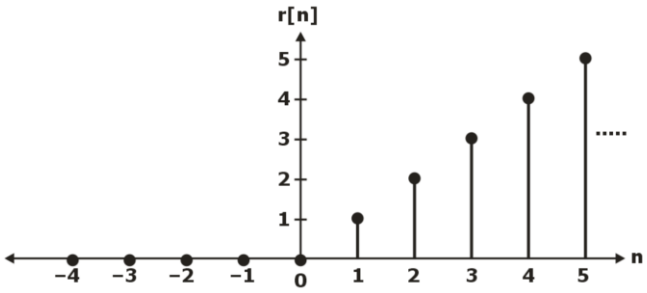



**Figure: DT unit Ramp Function**

## 13.4. Unit-Rectangular Function

The discrete-time unit rectangular sequence is shown in figure 31. It is defined as

$$rect\left[\frac{n}{2N}\right] = \begin{cases} 1, \ |n| \leq N \\ 0, \ |n| > N \end{cases}$$

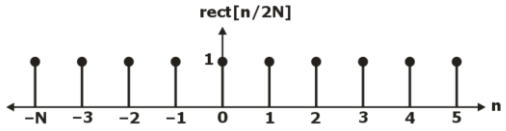



**Figure: DT unit rectangular function**

The signal rect [n/2N] has (2N + 1) unit samples over the interval – N ≤ n ≤ N.

## 13.5. Unit-Triangular Function

The discrete-time unit triangular sequence shown in figure 32 is defined as

$$tri\left(\frac{n}{N}\right) = \begin{cases} 1 - \frac{|n|}{N} & |n| \leq N \\ 0, & |n| > N \end{cases}$$

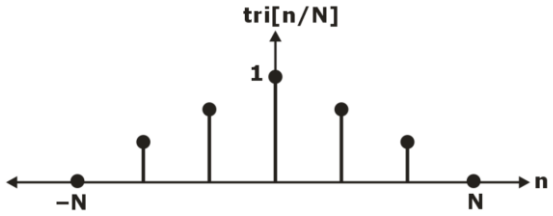



**Figure 1: DT unit Triangular Function**

The signal tri[n/N] has (2N + 1) unit samples over the interval – N ≤ n ≤ N.

### 13.6. Unit-Signum Function

The discrete-time function corresponding to the continuous time signum function is defined in figure 33.

$$sgn[n] = \begin{cases} 1, & n > 0 \\ 0, & n = 0 \\ -1, & n < 0 \end{cases}$$

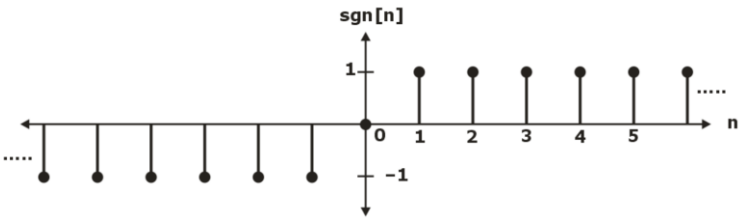



**Figure: DT unit Signum Function**

# CHAPTER 2: CLASSIFICATION OF SIGNALS

## 1. Periodic and Aperiodic signal

### 1.1 Condition for continuous-time periodic signal

$x(t + T) = x(t), \quad -\infty < t < \infty \quad \ldots(i)$

Where T is the fundamental period of a signal.

Frequency of the periodic signal is given by

$$f = \frac{1}{T}$$

Angular frequency, measured in radian per second, is defined as

$$\omega = \frac{2\pi}{T}$$

### 1.2 Condition for discrete-time periodic signal

$x(n + N) = x(n), \quad -\infty < n < \infty \ldots(ii)$

Where N is called the fundamental period of a signal.

The fundamental angular frequency or simply fundamental frequency of x[n] is given by,

$$\Omega = \frac{2\pi}{N} m$$

where, N = fundamental period

m = Smallest integer.

**Important point**

- The sum of two or more periodic discrete-time sequence is always periodic.
- A constant signal is periodic and its fundamental period is undefined.
- The sum of two or more periodic continuous-time signals need not be periodic. They will be periodic if and only if the ratio of their fundamental periods is rational.

### 1.3. Steps to determine whether the sum of two or more periodic signals is periodic or not.

**Step 1:** Determine the fundamental period of the individual signals in the sum signal, say $T_1$, $T_2$ ……

**Step 2:** Find the ratio of the fundamental period of the first signal with the fundamental periods of every other signals.

**Step 3:** If all the ratios are rational, then the sum signal is also periodic, and its fundamental period is

$$\frac{\text{LCM of Numerator of } T_1, T_2 \ldots\ldots}{HCF \text{ of Denominator of } T_1, T_2 \ldots\ldots}$$

### 1.4. Steps to determine whether the sum of two or more sequence periodic or not

**Step 1:** Determine the fundamental period of individual sequence in the sum sequence, say $N_1$, $N_2$ ….

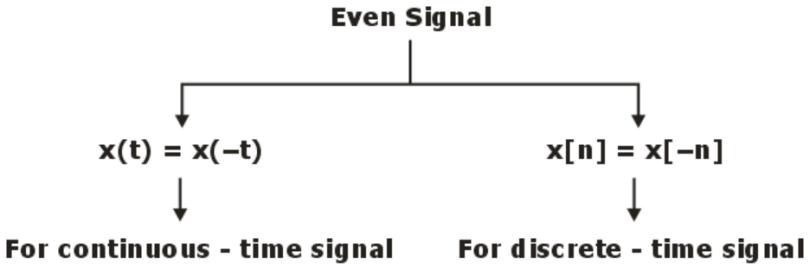
**Step 2:** If all the individual sequences are periodic then fundamental period is

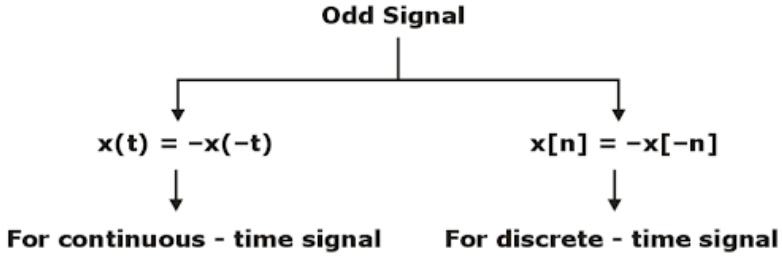
N = LCM of $N_1$, $N_2$, …..

**Step 3:** If any one or more sequence in sum is aperiodic then the resultant sequences are also aperiodic.

**2. Even and odd signals**

Even signals are symmetric about origin whereas odd signals are antisymmetric about origin.




An arbitrary signal x(t) can always be expressed as a sum of even and odd signals as

$x(t) = x_e(t) + x_o(t)$

Where, $x_e(t)$ is called the even part of x(t) and is given by

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

and $x_o(t)$ is called the odd part of x(t) and is given by

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$

**Basic properties**

- The sum of two even function is even and any constant multiple of an even function is even.
- The sum of two odd function is odd, and any constant multiple of an odd function is odd.
- The product of two even functions is an even function
- The product of two odd function is an even function.
- The product of an even function and an odd function is an odd function.
- Due to anti symmetry property, odd signal is always zero at t = 0

therefore, $x_o(0) = y_o(0) = 0$

or, $x_o[0] = y_o[0] = 0$

- Integration of a continuous-time odd signal within the limits [−T, T] results in a zero value ie.

$$\int_{-T}^{T} x_0(t)dt = \int_{-T}^{T} y_0(t)dt = 0$$

- The integral of a continuous-time even signal within the limits [−T, T] can be simplified as follow:

$$\int_{-T}^{T} x_e(t)dt = 2\int_{0}^{T} x_e(t)dt$$

- Adding the samples of discrete-time odd sequence $x_o[n]$ within the range [–N, N] is 0 ie,

$$\sum_{n=-N}^{N} x_o[n] = 0 = \sum_{n=-N}^{N} y_0[n]$$

- Adding the samples of discrete-time even sequence $x_e[n]$ within the range [–N, N] simplifies to

$$\sum_{n=-N}^{N} x_e[n] = x_e[0] + 2\sum_{n=1}^{N} x_e[n]$$

**Note:** Even and odd signals are mutually exclusive. That is, if a signal is an even signal, it cannot be odd and vice versa. however, there could be certain class of signals that could neither be termed odd nor even signal.

| s/g | s/g | s/g | |
|---|---|---|---|
| $x_1(t)x_2(n)$ | $x_2(t)x_2(n)$ | $x_1 \cdot x_2$ | $x_1/x_2$ |
| E | E → | E | E |
| E | O → | O | O |
| O | E → | O | O |
| O | O → | E | E |

"only for signals"

## 3. Energy and power signal

Signals which are classified on the basis of finite energy or finite average power are known as energy or power signal.

**Power signal**

Periodic signal is called power signal and for power signal, $P_\infty$ = finite & $E_\infty = \infty$.

Signal is referred to as power signal, if and only if the average power of the signal satisfies the condition

$0 < P < \infty$

**Energy signal**

For energy signal, $P_\infty = 0$ & $E_\infty$ = finite

Signal is referred as energy signal, if and only if the total energy of the signal satisfies the condition,

$0 < E < \infty$

**In case of Continuous-time signal**

Total energy is given by

$$E = \begin{cases} \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ \int_{-\infty}^{\infty} |x(t)|^2 dt \end{cases}$$

Average power is given by,

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

Average power of a periodic signal x(t) of fundamental period T is given by,

$$p = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

**In the case of discrete time signal**

Energy of signal is given as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Average power is defined by,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} |x[n]|^2$$

Average power in a periodic signal x(n) with fundamental period N is given by

$$p = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

**4. Energy and Power Continuous time Signals**

| | Energy Signal | | Power Signal |
|---|---|---|---|
| 1. | The total energy is obtained using $E = \lim_{T\to\infty} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$ | 1. | The average power is obtained using $P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$ |
| 2. | For the energy signal $0 < E < \infty$, and the average power $P = 0$. | 2. | For the power signal $0 < P < \infty$, and the energy $E = \infty$. |
| 3. | Non-periodic signals are energy signals. | 3. | Periodic signals are power signals. However, all power signals need not be periodic. |
| 4. | Energy signals are not time limited. | 4. | Power signals exist over infinite time. |

**Properties of Power signal:**

1) Power signal has infinite energy

2) $\boxed{\begin{array}{l} x(t) \longrightarrow P \\ x(-at + b) \longrightarrow P \end{array}}$

$$\begin{cases} A \sin t \longrightarrow \frac{A^2}{2} \\ A \sin (2t) \longrightarrow \frac{A^2}{2} \\ A \sin \left(2t - \frac{\pi}{6}\right) \longrightarrow \frac{A^2}{2} \end{cases}$$

3) K $x(at + b) \longrightarrow K^2 P$

**5. Causal and non-causal signal**

A continuous-time signal x(t) is said to be causal if x(t) = 0 for t < 0, otherwise the signal is non-causal. For an anti-causal signal x(t) = 0 for t > 0.

Similarly, a discrete-time signal x(n) is said to be causal if x(n) = 0 for n < 0; otherwise the signal is non- causal. For an anti-causal discrete-time signal x(n) = 0 for n > 0.

# CHAPTER 3: CLASSIFICATION OF SYSTEMS

## 1. Linear, nonlinear systems

A linear system is one which satisfies the principle of **superposition** and **homogeneity or scaling**.

Consider a linear system characterized by the transformation operator T[]. Let $x_1$, $x_2$ are the inputs applied to it and $y_1$, $y_2$ are the outputs. Then the following equations hold for a linear system

$$y_1 = T[x_1],\ y2 = T[x_2]$$

**Principle of homogeneity: $T[a*x_1] = a*y_1$, $T[b*x_2] = =b*y_2$**

**Principle of superposition: $T[x_1] + T[x_2] = a*y_1+b*y_2$**

**Linearity: $T[a*x_1] + T[b*x_2] = a*y_1+b*y_2$**

Where a, b are constants.

## 2. Time variant, time invariant systems

A system is said to be time variant system if its **response varies with time.** If the system response to an input signal does not change with time such system is termed as time invariant system. The behaviour and characteristics of time variant system are fixed over time.

In time invariant systems if input is delayed by time $t_0$ the output will also gets delayed by $t_0$. Mathematically it is specified as follows

$$\mathbf{y(t-t_0) = T[x(t-t_0)]}$$

For a discrete time invariant system the condition for time invariance can be formulated mathematically by replacing t as n*Ts is given as

$$\mathbf{y(n-n_0) = T[x(n-n_0)]}$$

Where $n_0$ is the time delay.

**Methodology**

1. Let $y(t, t_0)$ denotes the output corresponding to a delayed input $x(t -t_0)$. This can be obtained by substituting $x(t) \rightarrow x(t - t_0)$ in the given input-output relation

2. Now, obtain the delayed output $y(t - t_0)$, by directly substituting $t \rightarrow t - t_0$ in the given input-output relation

3. If $y(t, t_0) = y(t - t_0)$, then the system is time invariant. Otherwise it is a time-varying system

Similarly can be checked for discrete time signals also.

## 3. Systems with and without memory (Dynamic and Static systems)

A system is said to be static or memory less if its output at any instant depends on the input at that and does not depend on the past or future values of input. Otherwise, if the output at any

instant depends on the past or future values of input, then the system is said to be dynamic or with memory.

**4. Causal and Non-causal Systems**

A system is said to be causal, if the present value of the output signal depends only on the present value or past value or a combination of present and past values of the input signal. A system is said to be non-causal if it is not causal i.e., the present value of output depends on the future values of input. For example: (i) $y(t) = x(t) + x(t - 1)$ is a causal system.

(ii) The system $y[t] = x[n] + x[n — 1]$ is causal

**NOTE:** All memory less systems are causal systems because the output at any time instant depends only on the input at that time instant. Systems with memory can either be causal or non-causal.

**5. Invertible and Non-Invertible Systems**

A system is said to be invertible if the input to the system can be uniquely determined from the output. In order to have a system to be invertible, it is necessary that distinct inputs produce distinct outputs i.e., two different inputs cannot produce the same output.

If the system is invertible, there exists an inverse system. If these two systems are cascaded as shown in the figure, then final output is same as the input.

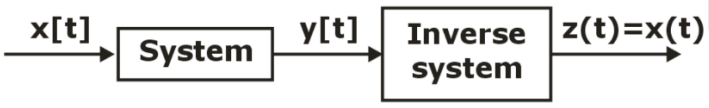



**Figure: CT Invertible System**

**Figure: DT Invertible System**

**NOTE: Invertible system**

A system is invertible if for the given to inputs x1(t) and x2(t) with $x1(t) \neq x2(t)$, it must be true that $y_1(t) \neq y_2(t)$

**6. Stable and Unstable systems**

A system is said to be bounded input and bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

The input signal x(t) is said to be bounded if there exists a finite number $M_x$ such that $|x(t)| \leq M_x < \infty$, for all t The system is BIBO stable if for any bounded input x(t) the output signal y(t) is also bounded i.e., $|y(t)| \leq M_y < \infty$, for all t. If the system produces unbounded output for bounded input then it is unstable.

A Discrete Time system is said to be BIBO stable if for any bounded input, it produces a bounded output.

The system is BIBO stable if for any bounded input x[n] the output signal y[n] is also bounded i.e.

$|y[n]| \leq M_y < \infty$, for all n

If the system produces unbounded output for a bounded input then it is unstable.

## 7. Properties of LTI systems in Terms of impulse Response

### 7.1. Memory less LTI system

A CT system is said to be memory less if its output at any time depends only on the value of the input at the same time. A memory less, linear time invariant system has an input output relation that is of the form

$y(t) = Kx(t)$

where K is any arbitrary constant. By substituting $x(t) = \delta(t)$ in equation, memory less LTI system has the impulse response

$h(t) = K\delta(t)$

**NOTE: Memory less LTI continuous system**

An LTI continuous system will be memory less if and only if its impulse response $h(t) = 0$ for $t \neq 0$.

### 7.2. Causal LTI System

An LTI system is said to be causal if the output at any instant depends only on the present and past values of the input. Consider a continuous-time LTI system whose output y(t) can be obtained using convolution integral given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

At t = 0 the output can be written as

$$y(0) = \int_{-\infty}^{\infty} h(\tau)x(-\tau)d\tau$$

For a causal system the output depends only on present and past values of input. From equation, we can see that, if $\tau \geq 0$, the output depends on present and past values of input and the system is causal. But if $\tau < 0$, then output depends on future values of input, therefore the system will be causal if $h(\tau) = 0$ for $\tau < 0$.

**NOTE:**

An LTI system will be causal if and only if its impulse response $h(t) = 0$

for $t < 0$.

### 7.3. Invertible LTI system

An LTI system is said to be invertible if the input of the system can be recovered from the output. As we discussed earlier. if the inverse system is connected in cascade with the original system, then final output will be same as the input. This can be illustrated in below figure.

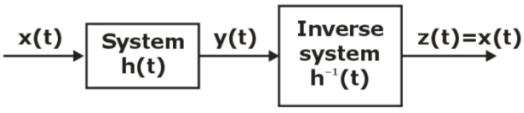



**Figure: An LTI inverse system**

**NOTE:**

An LTI system is invertible if its impulse response satisfies $h^{-1}(t) * h(t) = \delta(t)$

**NOTE:**

**An LTI system is BIBO stable if the impulse response is absolutely integrable**

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

## 8. Table Showing Comparison of Different Signals with Their Properties

| S.No. | Relationship between output and input | Linearity | Causality | Static or Dynamic | Time-Variancy |
|---|---|---|---|---|---|
| 1. | $y(t) = x(t - t_0)$ | Linear | Causal | Dynamic | Time-invariant |
| 2. | $y(t) = t.x(t)$ | Linear | Causal | Static | Time-variant |
| 3. | $y(t) = x(t) + A$ | Non-linear | Causal | Static | Time-invariant |
| 4. | $y(t) = x(\alpha t)$ | Linear | Non-causal | Dynamic | Time-variant |
| 5. | $y(t) = =x^2(t)$ | Non-linear | Causal | Static | Time-invariant |
| 6. | $y(t) = x(t^2)$ | Linear | Non-causal | Dynamic | Time-variant |
| 7. | $y(t) = \frac{dx(t)}{dt}$ | Linear | Causal | Dynamic | Time-invariant |
| 8. | $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ | Linear | Causal | Dynamic | Time-invariant |
| 9. | $y(t) = \int_{-\infty}^{at} x(\tau) d\tau$ | Linear | Non-causal | Dynamic | Time-variant |
| 10. | $y(t) = \cos[x(t)]$ | Non-linear | Causal | Static | Time-invariant |

| | | | | | |
|---|---|---|---|---|---|
| 11. | $y(t) = x(-t)$ | Linear | Non-causal | Dynamic | Time-variant |
| 12. | $y(t) = \cos\omega_0 t.x(t)$ | Linear | Causal | Static | Time-variant |

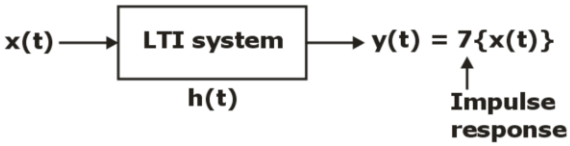
**Note:**

**⇒ All the static systems are causal systems but converse is not true.**

**⇒ All the non-causal systems are dynamic systems but the converse is not true.**

# CHAPTER 4: LTI SYSTEM (CONVOLUTION)

## 1. Convolution Integral




$$\boxed{y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

or

$$\boxed{y(t) = x(t) \otimes h(t)}$$

$$\boxed{u(t+\alpha) \otimes u(t+\beta) = r(t+\alpha+\beta)}$$

$$\boxed{y(t) = u(t) * r(t) = \frac{t^2}{2}u(t)} \; Parabolic function$$

$$\boxed{u(t+\alpha)| \otimes r(t+\beta)| = \frac{(t+\alpha+\beta)^2}{2}u(t+\alpha+\beta)}$$

## 2. Properties of convolution Integral

### i. Commutative property

$$\boxed{y(t) = x(t) \otimes h(t) = h(t) \otimes x(t)}$$

### ii. Distributive property

$x(t) \otimes [h_1(t) \otimes h_2(t)] = x(t) \otimes h_1(t) + x(t) \otimes h_2(t)$

### iii. Associative property

$[x(t) \otimes h_1(t)] \otimes h_2(t) = x(t) \otimes [h_1(t) \otimes h_2(t)]$

### iv. Property based on time invariancy

**a.** $x(t) \otimes h(t) = y(t)$

**b.** $x(t + \alpha) \otimes h(t) = y(t + \alpha)$

**c.** $x(t) \otimes h(t + \beta) = y(t + \beta)$

**d.** $x(t + \alpha) \otimes h(t + \beta) = y(t + \alpha + \beta)$

### v) Differentiation property:

If $x(t) * h(t) = y(t)$

Then,

$$\frac{dx(t)}{dt} * h(t) = \frac{dy(t)}{dt}$$

$$x(t) * \frac{dh(t)}{dt} = \frac{dy(t)}{dt}$$

$$\frac{dx(t)}{dt} * \frac{dh(t)}{dt} = \frac{d^2y(t)}{dt^2}$$

**vi.** When two equal width rectangular pulses of duration 'T' are convoluted, resultant signal is always a triangular pulse of duration "2T".

**vii.** When two unequal width rectangular pulses of duration $T_1$ and $T_2$ are convoluted, then the resultant signal is always a trapezoidal pulse of duration $T_1 + T_2$.

**viii. Invertibility of continuous time signals**

$$\boxed{h(t) = h_1(t) \otimes h_2(t) = \delta(t)}$$

**ix. Scaling property of convolution**

$$\boxed{x(at) \otimes h(at) = \frac{1}{a} y(at)}$$

**x. Stability**

For a continuous time LTI system to be stable, its impulse response should be absolutely integrable. i.e.

$$\boxed{\int_{-\infty}^{\infty} |h(t)| dt < \infty}$$

**xi. System in cascade connection**

$$\boxed{h_{cascade}(t) = h_1(t) \otimes h(t)}$$

**xii. System in parallel connection**

$$\boxed{h_{parallel}(t) = h_1(t) + h_2(t)}$$

## 3. Discrete Time LTI System

$$x[n] \rightarrow \boxed{LTI\ system} \rightarrow y[n] = x[n] \otimes h[n]$$

output response of a LTI system to an input x[n]

$$\boxed{y[n] = x[n] \otimes h[n] = \sum_{K=-\infty}^{\infty} x[K] h[n-K]}$$

convolution sum

## 4. Properties of convolution sum

**i. Commutative property**

$x[n] \otimes h[n] = h[n] \otimes x[n]$

**ii. Distributive property**

$x[n] \otimes (h_1[n] + h_2[n]) = x[n] \otimes h_1[n] + x[n] \otimes h_2[n]$

**iii. Associative property**

$x[n] \otimes h_1[n] \otimes h_2[n] = x[n] \otimes (h_1[n] \otimes h_2[n])$

**iv. Shifting property**

If $y[n] = x[n] \otimes h[n]$

then, $x[n - n_0] \otimes h\ n\ [n - n_1] = y[n - n_0 - n_1]$

**v. Duration of convolution**

Let M be the duration (length) of sequence x[n] and N be the duration (length) of sequence h[n], then the duration of convolution sum.

$y[n] = x[n] \otimes h[n]$ is **M+N-1**

**vi. Generalized Results**

**a.** If $x[n] = u(n)$

$h[n] = u(n)$

then $y[n] = x[n] \otimes h[n] = u[n] \otimes u[n] = (n + 1)\ u[n]$

**b.** $u[n + \alpha] \otimes u[n + \beta] = [n + \alpha + \beta + 1]\, u\, [n + \alpha + \beta]$

**vii. Systems in parallel**

$h[n] = h_1[n] + h_2[n]$

**viii. System in cascade**

$h[n] = h_1[n] \otimes h_2[n]$

# CHAPTER 5: CONTINUOUS TIME FOURIER SERIES (CTFS)

## 1. Existence of Fourier Series

The Fourier series for a periodic signal x(t) exists if it satisfies the following conditions which are knows as Dirichlet conditions:

- The function x(t) has a finite number of maxima and minima in one period.
- The function x(t) has a finite number of discontinuities in one period.
- The function x(t) is absolutely integrable over one period, that is,

$$\int_0^T |x(t)|dt < \infty$$

## 2. Fourier Series

| Fourier Series Form | Mathematical Expression | Coefficients |
|---|---|---|
| **Trigonometric** | $x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$ | $a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$<br>$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt$<br>$b_n = \frac{2}{T_0}\int_{T_0} x(t)(\sin n\omega_0 t)dt$ |
| **Exponential** | $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$ | $c_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$ |
| **Polar or Cosine Form** | $x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$ | $A_0 = a_0$<br>$A_n = \sqrt{a_n^2 + b_n^2}$<br>$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ |

**NOTE:** $c_n = \frac{1}{2}(a_n - jb_n)$,

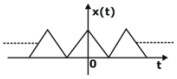
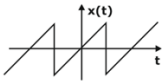
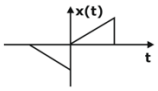
$c_{-n} = \frac{1}{2}(a_n + jb_n)$,.

## 3. Properties of Fourier Series

| S.N. | Property | CTFS Coefficients |
|---|---|---|
| 1 | Linearity | $px_1(t) + qx_2(t) \overset{CTFS}{\longleftrightarrow} pc_n + qd_n$ |
| 2 | Time shifting | $x(t - t_0) \overset{CTFS}{\longleftrightarrow} e^{-jn\omega_0 t_0}c_n$ |
| 3 | Time reversal | $x(-t) \overset{CTFS}{\longleftrightarrow} c_{-n}$ |
| 4 | Time scaling | $x(at) \overset{CTFS}{\longleftrightarrow} c_n$,<br>With period $aT_0$ |
| 5 | Multiplication | $x_1(t)x_2(t) \overset{CTFS}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{n-l}$ |

| | | |
|---|---|---|
| 6 | Conjugation and conjugate symmetry | $x*(t) \xleftrightarrow{CTFS} c_{-n}^*$ and $c_{-n} = c_n^*$ for x(t) is real |
| 7 | Time Differentiation | $\frac{dx(t)}{dt} \xleftrightarrow{CTFS} jn\omega_0 c_n$ |
| 8 | Time Integration | $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{CTFS} \frac{c_n}{jn\omega_0}$ |
| 9 | Convolution | $x_1(t) * x_2(t) \xleftrightarrow{CTFS} T_0 c_n d_n$ |
| 10 | Parseval's Theorem<br>If $x_1(t) = x_2(t) = x(t)$ | $\int_0^{T_0} x_1(t)x_2^*(t)dt = T_0 \sum_{n=-\infty}^{\infty} c_n d_n^*$<br>$\int_0^{T_0} \lvert x(t)^2 \rvert dt = T_0 \sum_{n=-\infty}^{\infty} \lvert c_n \rvert^2$ |
| 11 | Frequency Shifting | $e^{jk\omega_0 t}x(t) \xleftrightarrow{CTFS} c_{n-k}$ |

## 4. Condition for periodic signals to be symmetry

| Type of symmetry | Condition | Example | $a_0$ | $a_n$ | $b_n$ | Property |
|---|---|---|---|---|---|---|
| Even | x(t) = x(-t) |  | $a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$<br>$\Omega_0 = \frac{2\pi}{T}$<br>T is fundamental period | $\frac{4}{T}\int_0^{T/2} x(t)\cos(n\Omega_0 t)\,dt$ | 0 | Cosine terms only |
| Odd | x(t) = -x(-t) |  | 0 | 0 | $\frac{4}{T}\int_0^{T/2} x(t)\sin(n\Omega_0 t)\,dt$ | Sine terms only |
| Half wave | $x(t) = -x\left(t \pm \frac{T}{2}\right)$ |  | 0 | $\frac{4}{T}\int_0^{T/2} x(t)\cos(n\Omega_0 t)\,dt$ | $\frac{4}{T}\int_0^{T/2} x(t)\sin(n\Omega_0 t)\,dt$ | Odd n only |

# CHAPTER 6: CONTINUOUS TIME FOURIER TRANSFORM (CTFT)

## 1. Fourier Transform

Fourier transform is a transformation technique which transforms non-periodic signals from the continuous-time domain to the corresponding frequency domain. The Fourier transform of a continuous-time non periodic signal $x(t)$ is defined as

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

If the frequency is represented in terms of cyclic frequency *f* (in Hz), then the above equation is written as

$$X(jf) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

## 2. Existence of Fourier Transform

**Dirichlet Conditions**

(i) $x(t)$ is absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

(ii) $x(t)$ has a finite number of maxima and minima and a finite number of discontinuities within any finite interval.

## 3.MAGNITUDE AND PHASE SPECTRA

The Fourier transform $X(j\omega)$ of a signal $x(t)$ is in general, complex form can be expressed as

$$X(j\omega) = |X(j\omega)|\boxed{X(j\omega)}$$

The plot of $|X(j\omega)|$ versus $\omega$is called magnitude spectrum of *x(t)* and the plot of $\boxed{X(j\omega)}$versus $\omega$is called phase spectrum. The amplitude (magnitude) and phase spectra are together called Fourier spectrum which is nothing but frequency response of $X(j\omega)$ for the frequency range $-\infty < \omega < \infty$.

## 4. Inverse Fourier Transform

The inverse Fourier transform of $X(j\omega)$ is given as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega$$

## 5. Fourier Transform of Some Basic Signals

| S. No. | Time Domain $x(t)$ | Fourier Transform $X(j\omega)$ |
|---|---|---|
| 1. | $1$ | $2\pi\delta(\omega)$ |
| 2. | $\delta(t)$ | $1$ |
| 3. | $u(t)$ | $\pi\delta(\omega) + \frac{1}{j\omega}$ |
| 4. | $e^{-at}u(t)$ | $\frac{1}{a+j\omega}$ |
| 5. | $e^{-a\|t\|}$ | $\frac{2a}{a^2+\omega^2}$ |

| | | |
|---|---|---|
| 6. | $te^{-at}u(t)$ | $\frac{1}{(a+j\omega)^2}$ |
| 7. | $t^n e^{-at}u(t)$ | $\frac{n!}{(a+j\omega)^{n+1}}$ |
| 8. | $sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$ | $\frac{2}{j\omega}$ |
| 9. | $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ |
| 10. | $cos(\omega_0 t)$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ |
| 11. | $sin(\omega_0 t)$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ |
| 12. | $e^{-at}\, cos(\omega_0 t)\, u(t)$ | $\frac{a+j\omega}{(a+j\omega)^2 + \omega_0^2}$ |
| 13. | $e^{-at}\, sin(\omega_0 t)\, u(t)$ | $\frac{\omega_0}{(a+j\omega)^2 + \omega_0^2}$ |
| 14. | $\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \lvert t\rvert \le \tau/2 \\ 0 & \lvert t\rvert > \tau/2 \end{cases}$ | $\tau\, sin\, c\left(\frac{\omega\tau}{2\pi}\right)$ |
| 15. | $\frac{W}{\pi} sin\, c\left(\frac{Wt}{\pi}\right)$ | $\text{rect}\left(\frac{\omega}{2W}\right) = \begin{cases} 1 & \lvert\omega\rvert \le W \\ 0 & \lvert\omega\rvert > W \end{cases}$ |
| 16. | $\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{\lvert t\rvert}{\tau} & \lvert t\rvert \le \tau \\ 0 & Otherwise \end{cases}$ | $\tau\, sin\, c^2\left(\frac{\omega\tau}{2\pi}\right)$ |
| 17. | $\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$ | $\omega_0 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_0)$ |
| 18. | $e^{-t^2/2\sigma^2}$ | $\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$ |

## 6. Properties of Fourier Transform

| S. No. | Properly | Time Signal x(t) | Fourier Transform X(j $\omega$) |
|---|---|---|---|
| 1. | Linearity | $ax_1(t) + bx_2(t)$ | $aX_1(j\omega) + bX_2(j\omega)$ |
| 2. | Time Shifting | $X(t - t_0)$ | $e^{-j\omega t_0} X(j\omega)$ |
| 3. | Conjugation | $X^*(t)$ | $X^*(-j\omega)$ |
| 4. | Time Scaling | $X(at)$ | $\frac{1}{\lvert a\rvert} X\left(j\frac{\omega}{a}\right)$ |
| 5. | Differentiation in time | $\frac{d^n x(t)}{dt^n}$ | $(j\omega)^n X(j\omega)$ |
| 6. | Differentiation in frequency domain | $t\, x(t)$ | $j\frac{dX(j\omega)}{d\omega}$ |
| 7. | Time Integration | $\int_{-\infty}^{t} x(\tau)\, d\tau$ | $\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |
| 8. | Frequency Shifting | $X(t)\, e^{j\omega t}$ | $X[j(\omega - \omega_0)]$ |
| 9. | Duality | $X(t)$ | $2\pi x(-j\omega)$ |
| 10. | Time convolution | $X(t)*h(t)$ | $X(j\omega)\, H(j\omega)$ |
| 11. | Frequency Convolution | $x_1(t)x_2(t)$ | $\frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)]$ |
| 12. | Parseval's theorem | $E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$ | $E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$ |
| 13. | Time reversal | $X(-t)$ | $X(-j\omega)$ |

## 7. Important Points

(i) If $x(t)$ is a real and even symmetric function, then its Fourier transform $X(j\omega)$ is also real and even.

(ii) If $x(t)$ is real and odd symmetric signal, its Fourier transform $X(j\omega)$ is imaginary and odd symmetric.

(iii) If $x(t)$ is an imaginary and even symmetric function, then its Fourier transform $X(j\omega)$ is also imaginary and even symmetric.

(iv) If $x(t)$ is imaginary and odd symmetric signal, its Fourier transform $X(j\omega)$ is real and odd symmetric.

# CHAPTER 7: LAPLACE TRANSFORM

## 1. The Bilateral or Two-Sided Laplace Transform

The bilateral or two-sided Laplace transform of a continuous-time signal x(t) is defined as

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

## 2. The Unilateral Laplace Transform

The Laplace transform for causal signals and systems is referred to as the unilateral Laplace transform and is defined as follows:

$$X(s) = L\{x(t)\} = \int_{0}^{\infty} x(t)e^{-st}dt$$

**Comparison table for unilateral and bilateral Laplace transform:**

| Bilateral LT | Unilateral LT |
|---|---|
| 1. X(s) $\int_{-\infty}^{\infty} x(t)e^{-st}\,dt = LT[x(t)]$ | 1. $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt = ULT[x(t)]$ \ |
| 2. Limits of integration: $-\infty\ to\ +\infty$ | 2. Limits of integration: $0^-\ to\ \infty$ |
| 3. ROC is must | 3. No need to specify ROC (ROC must always be RHS of s- plane) |
| 4. BLT is unique if ROC is specified | 4.ULT is unique |
| 5. Handles both causal and non-causal systems | 5.Handles only causal systems |

## 3. THE EXISTENCE OF LAPLACE TRANSFORM

The bilateral Laplace transform of a signal x(t) exists if the following integral converges (i.e. finite)

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Substituting s = σ + jω in above equation

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty}[x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

The above integral converges if

$\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$

Hence, the Laplace transform of x(t) exists if x(t) $e^{-\sigma t}$ is absolutely integrable.

## 4. REGION OF CONVERGENCE

Laplace transform of x(t) i.e. X(s) exists if

$$\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$$

The range of values of σ (i.e. real part of s) for which the Laplace transform converges is known as Region of Convergence (ROC).

### 5. Laplace Transform of Some Basic Function

| S. No. | CT signal x(t) | Laplace Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ | ROC |
|---|---|---|---|
| 1. | δ(t) | $1$ | Entire s-plane |
| 2. | u(t) | $\frac{1}{s}$ | Re{s} > 0 |
| 3. | u(t) – u(t –a) | $\frac{1}{s}(1-e^{-as})$ | Re{s} > 0 |
| 4. | $e^{-at}$ u(t) | $\frac{1}{a+s}$ | Re {s} > – a |
| 5. | t u(t) | $\frac{1}{s^2}$ | Re {s} > 0 |
| 6. | $t^n$u(t) | $\frac{n!}{s^n+1}$ | Re {s} > 0 |
| 7. | $te^{-at}$ u(t) | $\frac{1}{(a+s)^2}$ | Re {s} > – a |
| 8. | $t^ne^{-at}$ u(t) | $\frac{n!}{(a+s)^{n+1}}$ | Re {s} > – a |
| 9. | cos(ω₀t)u(t) | $\frac{s}{\omega_0^2+s^2}$ | Re{s} > a |
| 10. | sin(ω₀t)u(t) | $\frac{\omega}{\omega_0^2+s^2}$ | Re {s} > 0 |
| 11. | x(t) = $\cos^2$(ω₀t)u(t) | $\frac{(2\omega_0^2+s^2)}{s(4\omega_0^2+s^2)}$ | Re {s} > 0 |
| 12. | x(t) = $\sin^2$(ω₀t) u(t) | $\frac{2\omega_0^2}{s(4\omega_0^2+s^2)}$ | Re {s} > 0 |
| 13. | x(t) = exp (–at) cos(ω₀t) u(t) | $\frac{a+s}{(a+s)^2+\omega_0^2}$ | Re {s} > – a |
| 14. | x(t)=exp(–at) sin(ω₀t) u(t) | $\frac{w_0}{(a+s)^2+w_0^2}$ | Re {s} > – a |

### 6. Properties of Laplace Transform

| S.N. | Property | Time function x(t) | ROC |
|---|---|---|---|
| 1. | Linearity | $ax_1(t)+bx_2(t) \overset{L}{\longleftrightarrow} aX_1(s)+bX_2(s)$ | At least $R_1 \cap R_2$ |
| 2. | Time scaling | $x(at) \overset{L}{\longleftrightarrow} \frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$ | $aR_x$ |
| 3. | Time shifting | $x(t-t_0) \overset{L}{\longleftrightarrow} e^{-st_0}X(s)$ | $R_x$ |
| 4. | Frequency shifting | $e^{s_0t}x(t) \overset{L}{\longleftrightarrow} X(s-s_0)$ | $R_x$ + Re($s_0$) |

| | | | |
|---|---|---|---|
| 5. | Time differentiation | $\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s) - x(0)$ | $R_x$ |
| 6. | time integration | $\int_0^t x(\tau)\, d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}$ | R ∩ Re(s) >0 |
| 7. | s-domain differentiation | $-tx(t) \overset{L}{\longleftrightarrow} \frac{dX(s)}{ds}$ | $R_x$ |
| 8. | Conjugation | $x^*(t) \overset{L}{\longleftrightarrow} X^*(s*)$ | $R_x$ |
| 9. | Time convolution | $x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s)X_2(s)$ | atleast $R_1 \cap R_2$ |
| 10. | s-domain convolution | $x_1(t)x_2(t) \overset{L}{\longleftrightarrow} \frac{1}{2\pi j}[X_1(s) * X_2(s)]$ | atleast $R_1 \cap R_2$ |
| 11. | Initial value theorem | $x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$ | |
| 12. | Final value theorem | $x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$ | |
| 13. | Time Reversal | $x(-t) \overset{L}{\longleftrightarrow} X(-s)$ | $-R_x$ |

## 7.IMPULSE RESPONSE AND TRANSFER FUNCTION

Let $x(t) \overset{L}{\longleftrightarrow} X(s)$ is the input and $y(t) \overset{L}{\longleftrightarrow} Y(s)$ is the output of an LTI continuous time system having impulse response $h(t) \overset{L}{\longleftrightarrow} H(s)$. The response y(t) of the continuous time system is given by convolution integral of input and impulse response as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Using convolution property of Laplace transform the above equation can be written as.

Y(s) = X (s) H (s)

Thus $H(s) = \frac{Y(s)}{X(s)}$

Where, H(s) defined as the transfer function of the system. It is the Laplace transform of the impulse response.

Impulse response is

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{Y(s)}{X(s)}\right\}$$

## 8. STABILITY AND CAUSALITY

For a causal system the ROC of its rational transfer function H(s) is to the right of the righter most pole.

For the system to be stable (i.e. the ROC of its system function H(s) includes the entire jω-axis) the righter most pole of H(s) must be to the left of jω axis.

**NOTE:** A causal system with rational transfer function H(s) is stable is and only if all its poles lie in the negative half of s-plane.

## 9. SYSTEM FUNCTION FOR INTERCONNECTED LTI SYSTEMS

### 1. Parallel Connection

The parallel interconnection of two LTI continuous systems having impulse responses $h_1(t)$ and $h_2(t)$ is shown in the below figure.

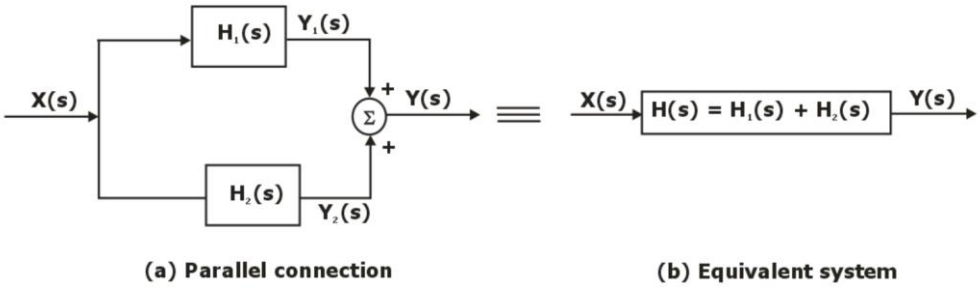



**Figure: Parallel connection of LTI system in s-domain**

### 2. Cascaded Connection

Two systems with impulse responses $h_1(t)$ and $h_2$ are connected in cascaded configuration as shown in below figure.

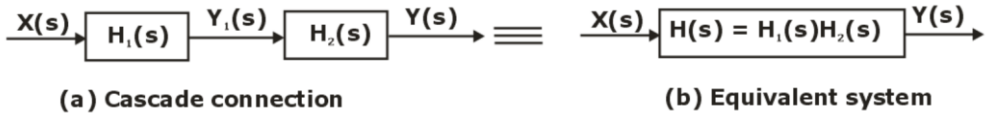



**Figure: Cascaded connection of LTI system in s-domain**

## 10. ZERO INPUT RESPONSE AND ZERO-STATE RESPONSE

The Laplace transform gives total response which includes zero input response and zero state response components.

Total response = Zero input response + Zero state response

### 10.1 Zero- input response:

The input is considered as zero and response is due to the initial conditions i.e. initial conditions generates the output.

### 10.2 Zero state response:

The initial conditions are considered as zero (i.e. zero state) and response is due to applied input. The term zero state signifies the system is initially released.

This is also termed as forced response as we are applying input (i.e., force to the system) with zero initial condition.

# CHAPTER 8: Z TRANSFORM

## 1. The Bilateral or Two-sided Z-transform

The z-transform of a discrete time sequence x[n], is defined as

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

## 2. The unilateral or One-sided z-transform

The z-transform for causal signals and systems is referred to as the unilateral z-transform. For a causal sequence

z[n] = 0, for n < 0

Therefore, the unilateral z-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

## 3. EXISTENCE OF Z-TRANSFORM

For existence of z-transform

|X(z)| < ∞

$$\sum_{n=-\infty}^{\infty} x[n]r^{-n} < \infty$$

## 4. Standard z transforms with their respective ROCs.

| S.No. | DT sequence x[n] | z-transform | ROC |
|---|---|---|---|
| 1. | δ[n] | 1 | Entire z-plane |
| 2. | δ [n – $n_0$] | $Z^{-n0}$ | Entire z-plane except z = 0 |
| 3. | u[n] | $\frac{1}{1-z^{-1}} = \frac{z}{z-1}$ | \|z\| > 1 |
| 4. | $a^n$u[n] | $\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$ | \|z\| > \|a\| |
| 5. | $a^{n-1}$u[n – 1] | $\frac{z^{-1}}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$ | \|z\| > \|a\| |
| 6. | nu[n] | $\frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$ | \|z\| > 1 |
| 7. | n$a^n$u[n] | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{\alpha z}{(z-\alpha)^2}$ | \|z\| > a |
| 8. | cos ($\Omega_0$n) u[n] | $\frac{1-z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}$ or $\frac{z[z-\cos\Omega_0]}{z^2-2z\cos\Omega_0+1}$ | \|z\| > 1 |
| 9. | sin($\Omega_0$n) u[n] | $\frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}$ or $\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$ | \|z\| > 1 |
| 10. | $a^n$ cos($\Omega_0$n)u[n] | $1-\alpha z^{-1}\cos\Omega_0$ | \|z\| > \|a\| |

| | | | |
|---|---|---|---|
| | | $\frac{z[z-\alpha\cos\Omega_0]}{z^2-2\alpha z\cos\Omega_0+\alpha^2}$ | |
| 11. | $\alpha^n \sin(\Omega_0 n)\, u[n]$ | $\frac{\alpha z^{-1}\sin\Omega_0}{1-2\alpha z^{-1}\cos\Omega_0+\alpha^2 z^{-2}}$ or $\frac{\alpha z\sin\Omega_0}{z^2-2\alpha z\cos\Omega_0+\alpha^2}$ | $\lvert z\rvert > \alpha$ |
| 12. | $r\,\alpha^n \sin(\Omega_0 n+\theta)\, u[n]$ with $\alpha \in R$ | $\frac{A+Bz^{-1}}{1+2\gamma z^{-1}+\alpha^2 z^{-2}}$ or $\frac{z(Az+B)}{z^2+2\gamma z+\gamma^2}$ | $\lvert z\rvert \le \lvert\alpha\rvert^{(n)}$ |

## 5. Properties of z- transform

| Properties | Time domain | z-transform | ROC |
|---|---|---|---|
| Linearity | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | at least $R_1 \cap R_2$ |
| Time shifting (bilateral or non-causal) | $x[n - n_0]$ | $Z^{-n_0}X(z)$ | $R_x$ except for the possible deletion or addition of z = 0 or z = ∞ |
| | $x[n + n_0]$ | $Z^{n_0}X(z)$ | |
| Time shifting (unilateral or causal) | $x[n - n_0]$ | $z^{-n_0}\left(X(z)+\sum_{m=1}^{n_0} x[-m]z^{m}\right)$ | $R_x$ except for the possible deletion or addition of z = 0 or z = ∞ |
| | $x[n + n_0]$ | $z^{n_0}\left(X(z)-\sum_{m=1}^{n_0-1} x[m]z^{-m}\right)$ | |
| Time reversal | $x[-n]$ | $X\left(\frac{1}{z}\right)$ | $1/R_x$ |
| Differentiation in z domain | $nx[n]$ | $-z\frac{dX(z)}{dz}$ | $R_x$ |
| Scaling in z domain | $a^n x[n]$ | $X\left(\frac{z}{a}\right)$ | $\lvert a\rvert R_x$ |
| Time scaling (expansion) | $x_k[n] = x[n/k]$ | $X(z^k)$ | $(R_x)^{1/k}$ |
| Time differencing | $x[n] - x[n - 1]$ | $(1 - z^{-1})\, X(z)$ | $R_x$, except for the possible deletion of the origin |
| Time convolution | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | at least $R_1 \cap R_2$ |
| Conjugations | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |

| | | | |
|---|---|---|---|
| Initial-value theorem | | $x[0] = \lim_{z\to\infty} X(z)$ | provided x[n] = 0 for n < 0 |
| Final-value theorem | | $x[\infty] = \lim_{n\to\infty} x(n) = \lim_{x\to 1}(z-1)X(z)$ | provided x[∞] exists |

**6.Causality**

A linear time-invariant discrete time system is said to be causal if the impulse response **h[n] = 0, for n < 0** and it is therefore **right-sided.** The ROC of such a system H(z) is the exterior of a circle. If H(z) is rational then the system is said to be causal if

1. The ROC is the exterior of a circle outside the outermost pole; and

2. The degree of the numerator polynomial of H9z) should be less than or equal to the degree of the denominator polynomial.

**NOTE:** $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ condition for the stability.

# CHAPTER 9: DISCRETE TIME FOURIER TRANSFORM (DTFT)

## 1. DTFT

The DTFT of a non-periodic sequence x[n] is given by

$X(e^{j\Omega}) = F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$

## 2. Magnitude and Phase Spectra

The Fourier transform $X(e^{j\Omega})$ of x[n] is, in general, complex and can be expressed as

$$X(e^{j\Omega}) = |X(e^{j\Omega})| \angle X(e^{j\Omega})$$

The plot of $|X(e^{j\Omega})|$ versus Ω is called magnitude spectrum of x[n] and the plot of $\angle X(e^{j\Omega})$ versus Ω is called phase spectrum. The amplitude (magnitude) and phase spectra are together called Fourier spectrum of signal x[n].

## 3. Existence of DTFT

The DTFT $X(e^{j\Omega})$ of a DT sequence x[n] exists, if x[n] is absolutely summable i.e.

$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

## 4. Inverse DTFT

The inverse discrete time Fourier transform of $X(e^{j\Omega})$ is defined as:

$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$

## 5. Special forms of DTFT

| Discrete Sequence x[n] | DTFT $X(e^{j\Omega})$ |
|---|---|
| Real and Even | Real and Even |
| Real and Odd | Imaginary and Odd |
| Imaginary and Even | Imaginary and Even |
| Imaginary and Odd | Real and Odd |

## 6. DTFT of some basic DT signals

| Signal | Fourier Transform |
|---|---|
| 1 | $2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$ |
| δ[n] | 1 |
| u[n] | $\frac{1}{1-e^{-j\Omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$ |
| $a^n u[n], \lvert a\rvert < 1$ | $\frac{1}{1-ae^{-j\Omega}}$ |

| | |
|---|---|
| $\delta[n-n_0]$ | $e^{-j\Omega n_0}$ |
| $e^{-j\Omega_0 n}$ | $2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega-\Omega_0-2\pi k)$ |
| $cos\ \Omega_0\ n$ | $\pi \sum_{k=-\infty}^{\infty} \{\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)\}$ |
| $sin\ \Omega_0\ n$ | $\frac{\pi}{j} \sum_{k=-\infty}^{\infty} \{\delta(\Omega-\Omega_0-2\pi k)-\delta(\Omega+\Omega_0-2\pi k)\}$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n-kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega-\frac{2\pi k}{N}\right)$ |
| $(n+1)a^n u[n]$ $\lvert a\rvert < 1$ | $\frac{1}{(1-ae^{-j\Omega})^2}$ |

## 7. Properties of DTFT

| S. No. | Property | Sequence | DTFT |
|---|---|---|---|
| 1. | Linearity | $a_1x_1[n] + a_2x_2[n]$ | $a_1X_1(e^{j\Omega}) + a_2X_2(e^{j\Omega})$ |
| 2. | Periodicity | $x[n]$ | $X(e^{j\Omega}) = X(e^{j(\Omega+2n)})$ |
| 3. | Time shifting | $x[n - k]$ | $e^{-j\Omega k} X(e^{j\Omega})$ |
| 4. | Frequency shifting | $e^{j\Omega_0 n}x[n]$ | $X(e^{j(\Omega-\Omega_0)})$ |
| 5. | Time reversal | $x[-n]$ | $X(e^{-\Omega})$ |
| 6. | Time expansion | $X[n/k]$, n is a multiple integer of k | $X(e^{jk\Omega})$ |
| 7. | Differentiation in the frequency domain | $nx[n]$ | $j\frac{d}{d\Omega}X(e^{j\Omega})$ |
| 8. | Conjugation | $x^*[n]$ | $X^*(e^{-j\Omega})$ |
| 9. | Convolution in time Domain | $x_1[n] * x_2[n]$ | $X_1(e^{j\Omega})\ X_2(e^{j\Omega})$ |
| 10. | Convolution in the frequency domain | $x_1[n]x_2[n]$ | $\frac{1}{2\pi}[X_1(e^{j\Omega}) * X_2(e^{j\Omega})]$ |
| 11. | Time differencing | $x[n] - x[n - 1]$ | $(1 - e^{-j\Omega})\ X(e^{j\Omega})$ |
| 12. | Time accumulation | $\sum_{n=-\infty}^{\infty} x[n]$ | $\frac{1}{(1-e^{-j\Omega})}X(e^{j\Omega})$ $+\pi X(0) \sum_{m=-\infty}^{\infty} \delta(\Omega-2\pi m)$ |
| 13. | Parseval's Theorem | energy of the sequence x[n] is given as $\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\Omega})\rvert^2 d\Omega$ | |

### 8. Transfer Function & Impulse Response

Let $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ be the input sequence and $y[n] \xleftrightarrow{DTFT} Y(e^{j\Omega})$ be the output sequence of the system. Then, response (output) of the system is given by following convolution sum.

$$y[n] = x[n] * h[n]$$

by taking DTFT using the convolution property, we have

$Y(e^{j\Omega}) = X(e^{j\Omega})\ H(e^{j\Omega})$

$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$ (Transfer function)

**NOTE:** $X(e^{j\Omega})$ exists if the ROC of X(z) includes the unit circle. We know that ROC does not contain any poles. Therefore, it is concluded that DTFT of any sequence x[n] can be obtained from its z-transform X(z) if the poles of X(z) are inside the unit circle.

# CHAPTER 10: SAMPLING THEOREM

## 1. Sampling

It is the process of converting a continuous time-signal x(t) into a discrete time signal x[n] by taking samples of the continuous time signal at discrete intervals of time. That is,

$$x[n] = x(t)|_{t=nT_s}$$

Where, n is any integer.

## 2. SAMPLING THEOREM

The sampling theorem states that, a band-limited signal x(t) having the highest frequency component $f_m$ Hz can be exactly recovered or reconstructed from its samples taken at a rate of $2f_m$ samples per second. Hence, the sampling rate or sampling frequency must satisfy this condition,

$$f_s \geq 2f_m$$

The sampling interval

$$T_s = \frac{1}{f_s} \leq \frac{1}{2f_m}$$

# CHAPTER 11: DISCRETE FOURIER TRANSFORM (DFT)

## 1. Discrete Fourier Transform

The discrete-time Fourier transform of the sequence x[n] is given by

$$X_{DFT}[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}$$

where k = 0, 1, 2, …., (N – 1)

Discrete Fourier transform $X_{DFT}[k]$ is also denotes as X[k] or DFT{x[n]}

## 2. INVERSE DISCRETE FOURIER TRANSFORM (IDFT)

The inverse Discrete Fourier transform of $X_{DFT}(k)$ is defined as

$$x[n] = \frac{1}{N} \sum_{k\,=\,0}^{N-1} X[k] e^{j2\pi nk/N}$$

## 3. Properties of DFT

| Discrete Sequence x[n] | DFT X[k] |
|---|---|
| Real and Even | Real and Even |
| Real and Odd | Imaginary and Odd |
| Imaginary and Even | Imaginary and Even |
| Imaginary and Odd | Real and Odd |

| S. No. | Properties | Discrete time Signal | Discrete Fourier Transform |
|---|---|---|---|
| 1 | Linearity | $ax_1[n] + bx_2[n]$ | $aX_1[k] + bX_2[k]$ |
| 2 | Periodicity | x[n + N] = x[n] | X[k + N] = X[k] |
| 3 | Circular time shift | $x[n - n_0]_N$ | $X(k)e^{\frac{-j2\pi kn_0}{N}}$ |
| 4 | Time reversal | x[N – n] | X(N – k) |
| 5 | Conjugation | x*[n] | X*[N – k] |
| 6. | Circular frequency shift | $x[n]e^{\frac{j2\pi k_0 n}{N}}$ | $X[(k - k_0)]_N$ |
| 7. | Multiplication | $x_1[n]\ x_2[n]$ | $\frac{1}{N}(X_1[k] * X_2[k])$ |
| 8 | Circular Convolution | $x_1[n] \circledast x_2[n]$ | $X_1[k]\ X_2[k]$ |
| 9 | Circular correlation | $\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x[n] y * [n-m]_N$ | $X(k)Y^*(k)$ |
| 10 | Parseval's relation | $\sum_{n=0}^{N-1} x_1[n] x_2^*[n]$ | $\frac{1}{N}\sum_{k=0}^{N-1} X_1[k] X_2^*[k]$ |

# CHAPTER 12: FAST FOURIER TRANSFORM (FFT)

## 1. Properties of Twiddle factor:

### (i) Symmetry property:

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

### (ii) Periodicity property:

$$W_N^{k+N} = W_N^k$$

**Note:** The fast Fourier transform algorithm exploits the two basic properties of the twiddle factor and reduces the number of complex multiplications required to perform DFT from $N^2$ to $\frac{N}{2} log_2 N$.

## 2. IDFT USING FFT ALGORITHM

FFT algorithm can be used to compute an inverse DFT without any change in the algorithm. The inverse DFT of an N-point sequence X(k), k = 0, 1, …., N – 1 is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-nk}$$

Where, $W = e^{-j2\pi/N}$

Take complex conjugate and multiply by N, we obtain

$$\boldsymbol{Nx*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X*(k)W^{-nk}}$$

The desired output sequence x(n) can then found by complex conjugating the DFT and dividing by N to give

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X*(k)W^{-nk}\right]^*$$

# CHAPTER 13: DIGITAL FILTERS

**1.** There are two class of digital filters, depending on the duration of the impulse response.

**1.(i) Finite-duration impulse response (FIR) digital filter,** the operation of which governed by linear constant coefficient difference equations of a non-recursive nature.

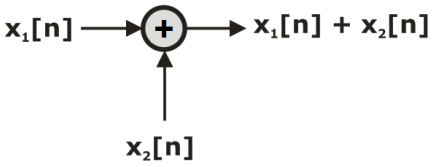
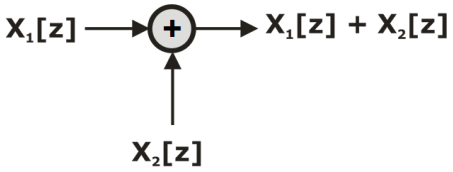
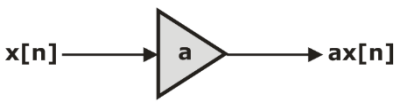
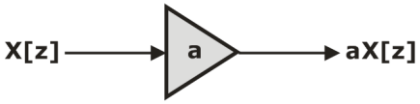
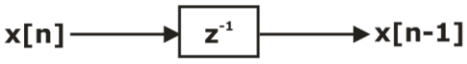
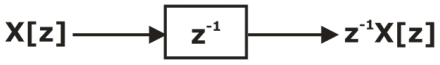
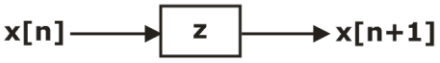
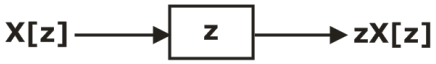
The transfer function of FIR digital filter is a polynomial in $z^{-1}$.

It has three important properties.

- They have finite memory and therefore, any transient start up is of limited duration.
- They are always BIBO stable.
- They can realize a desired magnetic response with exactly linear phase (i.e. with no phase distortion.)

**1.(ii) Infinite-duration impulse response (IIR) digital filter**, whose input output characteristics are governed by linear constant coefficient difference equations of recursive nature.

**2. Basic Elements of Block Diagram**

| Elements of Block diagram | Time Domain Representation | s-domain Representation |
|---|---|---|
| Adder |   |   |
| Constant multiplier |   |   |
| Unit delay element |   |   |
| Unit advance element |   |   |

## 3. COMPARISION BETWEEN FILTERS

**Table 2: Comparison Between Non-Recursive & Recursive filter**

| Non-Recursive filters | Recursive filters |
|---|---|
| $y(n) = \sum_{k=-\infty}^{\infty} a_k x(n-k)$<br>for causal system<br>$= \sum_{k=0}^{\infty} a_k x(n-k)$<br>For causal i/p sequence<br>$y(n) = \sum_{k=0}^{N} a_k x(n-k)$<br>It gives FIR output. All zero filter.<br>It is always stable. | $y(n) = \sum_{k-N_f}^{N_p} a_k x(n-k) - \sum_{k=1}^{M} b_k y(n-k)$<br>for causal system<br>$y(n) = \sum_{k_0}^{N_p} a_k x(n-k) - \sum_{k=1}^{M} b_k y(n-k)$<br>It gives IIR output but not always.<br>Ex: y(n) = x(n) – x(n – 3) + y(n – 1)<br>General TF: $H(z) = \frac{\sum_{k=-N_F}^{N_p} a_k z^{-k}}{1-\sum_{k=1}^{M} b_k z^{-k}}$<br>$b_k$ = 0 for Non-Recursive<br>$N_f$ = 0 for causal system |

## 4. Comparison Between FIR & IIR Filters

| FIR filters | IIR filters |
|---|---|
| 1. Linear phase no phase distortion. | Linear phase, phase distortion. |
| 2. Used in speech processing, data transmission & correlation processing. | Graphic equalizers for digital audio, tone generators filters for digital telephone |
| 3. Realized non recursively. | Realized recursively. |
| 4. Stable | Stable or unstable.<br>H(n) = $a^n$(n), a < 1 stable<br>= 0, a > 1 unstable |
| 5. Filter order is more | Less |
| 6. More co-efficient storage | Less storage |
| 7. Quantization noise due to finite precision arithmetic can be made negligible | Quantization noise |
| 8. Co-efficient accuracy problem is less severe. | More |
| 9. Used in multi rate DSP (variable sampling rate) | |

****

BYJU'S EXAM PREP

# Electrical Engineering

## Electromagnetic Theory

SHORT NOTES

# IMPORTANT FORMULAS TO REMEMBER

## CHAPTER 1: VECTOR CALCULUS

### 1. Vector Quantity

A physical quantity which has both magnitude and definite direction is called a vector quantity. The various examples of vector quantity are force, velocity, displacement, electric field intensity, magnetic field intensity, acceleration etc.

#### 1.1. Representation of a Vector

To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as $\bar{a}$ and $\bar{b}$, or by a letter in boldface type such as $\bar{\mathrm{A}}$ and $\bar{\mathrm{B}}$.

#### 1.2. Unit Vector

A unit vector consists both magnitude and direction. Its direction is same as that of the main vector however, its magnitude is unity. It can be written in various as $I_A$, $i_A$, $a_A$ or $u_A$. A unit vector is defined as the ratio of the main vector itself to its magnitude. For example, the unit vector of A is given as $\alpha_A = \dfrac{\bar{A}}{|A|}$

Where |A| is the magnitude of the vector and $a_A$ is the unit vector of A.

### 2. Basic Vector Operations

#### 2.1. Scaling of a Vector

When a vector is multiplied by a scalar it results in a vector quantity.

Consider a vector $\vec{\mathrm{A}}$ and a scalar k. The product $\vec{\mathrm{R}}$ of the two quantities is given as

$\vec{\mathrm{R}} = \mathrm{k}\vec{\mathrm{A}}$

Following are some important properties of scaling operation:

**Properties of scaling operation**

**1.** Consider the scaling operation $\vec{\mathrm{R}} = \mathrm{k}\vec{\mathrm{A}}$. The direction of $\vec{\mathrm{R}}$ is same as that of $\vec{\mathrm{A}}$ if k is positive, and opposite to that of A if k is negative.

**2.** In rectangular coordinates, assume that the scaling operation is given by

$R_x \bar{a}_x + R_y \bar{a}_y + R_Z \bar{a}_Z = k\,(A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_Z)$

The above equality is satisfied if each component of the LHS is equal to the corresponding component of RHS, i.e.

$R_x = kA_x$, $R_y = kA_y$, $R_z = kA_z$

The magnitude of R is

$$|R| = k\sqrt{A_x^2 + A_y^2 + A_z^2} = kA$$

**3.** Let $k_1$, $k_2$ be the scalars, and A, B be the vectors then,

$(k_1+k_2)\ \vec{\mathrm{A}} = k_1\vec{\mathrm{A}} + k_2\vec{\mathrm{A}}$

### 2.2. Addition of Vectors

Consider the two vectors,

$\vec{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_Z$ and

$\vec{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_Z$

The addition of these two vectors is given by

$\vec{A} + \vec{B} = (A_x+B_x)\ \bar{a}_x + (A_y+B_y)\ \bar{a}_y + (A_Z+B_Z)\ \bar{a}_Z$

**Properties of Vector Addition**

**1.** Vector addition follows the commutative law, i.e.

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

**2.** Vector addition follows the associative law, i.e.

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

**3.** Similar to the vector addition, the subtraction of the vectors is defined as

$\vec{A} - \vec{B} = (A_x - B_x)\ \bar{a}_x + (A_y - By)\ \bar{a}_y + (A_z - B_Z)\ \bar{a}_Z$

$k_1(\vec{A} + \vec{B}) = k_1\vec{A} + k_1\vec{B}$

### 2.3. Multiplication of vectors

When two vectors A and B are multiplied, the result is either a scalar or a vector depending on how they are multiplied. There are two types of vector multiplication:

1. Scalar (or dot) product: $\vec{A} \bullet \vec{B}$

2. Vector (or cross) product: $\vec{A} \times \vec{B}$

#### 2.3.1 Scalar Product

The dot product of the vectors A and B is defined as

$\vec{A} \bullet \vec{B} = |A||B| \cos\theta$

Following are some important properties of dot product of two vectors.

**Properties of Dot product**

**1.** The dot product of two orthogonal vectors is always zero, i.e.

$\vec{A} \bullet \vec{B} = 0$, if $\theta = 90^0$

**2.** The dot product of two parallel vectors in equal to the product of their magnitudes, i.e.

$\vec{A} \bullet \vec{B} = AB$, if $\theta = 0^0$

**3.** In rectangular coordinate systems, the dot products of the unit vectors are given as

$a_x \bullet a_y = a_y \bullet a_z = a_z \bullet a_x = 0$

$a_x \bullet a_x = a_y \bullet a_y = a_z \bullet a_z = 1$

**4.** if the two vectors are defined in rectangular coordinates as

$\vec{A} = A_x a_x + A_y a_y + A_z a_z$

$\vec{B} = B_x a_x + B_y a_y + B_z a_z$

Then, their dot product is evaluated as

$\vec{A} \bullet \vec{B} = A_xB_x + A_yB_y + A_zB_z$

**5**. The dot product follows the commutative law, i.e.

$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

**6.** The dot product also follows the distributive law, i.e.

$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

### 2.3.2 Vector of Cross Product

The cross product of two vectors A and B is Defined as

$\vec{A} \times \vec{B} = (AB \sin \theta)\, a_n$

Where $a_n$ is the unit vector normal to the plane containing A and B, θ is the angle between the vector A and B as shown in Figure 1.4 As there are the normal unit vector $a_n$ we use the right- hand rule.

**Right hand rule**

Let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward second; then your thumb indicates the direction of $a_n$. the cross-product A× B points upward.

**Properties of cross product**

**1.** The cross product of two orthogonal vectors is equal to the product of their magnitudes with the direction perpendicular to the plane, i.e.

$\vec{A} \times \vec{B} = AB a_n, \quad$ if, $\theta = 90^0$

**2.** The cross product of two parallel vectors is always zero, i.e.

$\vec{A} \times \vec{B} = 0, \quad$ if $\theta = 0^0$

**3.** In rectangular coordinate system, the cross product of the unit vectors are given as

$a_x \times a_x = a_y \times a_y = a_z \times a_z = 0$

$a_x \times a_y = a_z$

$a_y \times a_z = a_x$

$a_z \times a_x = a_y$

**4.** If the two vectors are defined in rectangular coordinates as

$\vec{A} = A_xa_x + A_ya_y + A_za_z$

$\vec{B} = B_xa_x + B_ya_y + B_za_z$

Then, their cross product is evaluated as

$$\vec{A} \times \vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

**5.** The cross product is anti-commutative, i.e.

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

**6.** The cross product follows the distributive law, i.e.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

## 3. COORDINATE SYSTEMS

The following three most useful coordinate systems:

1. Cartesian or rectangular coordinates,
2. Circular or cylindrical coordinates, and
3. Spherical coordinates.

### 3.1. Rectangular or Cartesian Coordinate System

The three coordinate axes are designated as x, y and z which are mutually perpendicular to each other. The variables x, y and z can have any values in the range

$-\infty < x < \infty$, $-\infty$, $< y < \infty$, $-\infty < z < \infty$

Vector Representation in Rectangular Coordinate System

A vector A in rectangular coordinate system is represented as

$A = A_x\, a_x + A_y\, a_y + A_z\, a_z$

Where $a_x, a_y, a_z$ are the unit vectors along the x, y and z directions.

The magnitude of A is given by

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

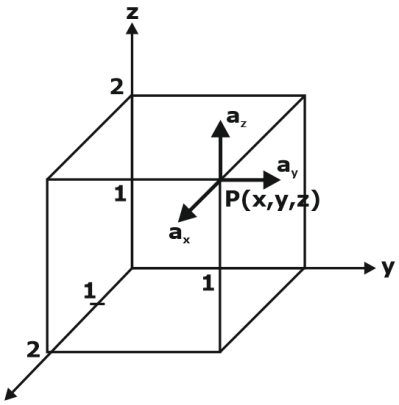



**Figure: Representation of cartesian coordinates**

### 3.2. Cylindrical Coordinates

The cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A Point P in cylindrical coordinates is represented as ($\rho$, $\phi$, z) and is as shown in figure below.

The ranges of the variables are:

$0 \le \rho \le \infty$

$0 \le \phi \le 2\pi$

$-\infty \le z \le +\infty$

A vector $\vec{A}$ in cylindrical coordinates can be written as

$\left(A_\rho, A_\phi, A_z\right)$ or $A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

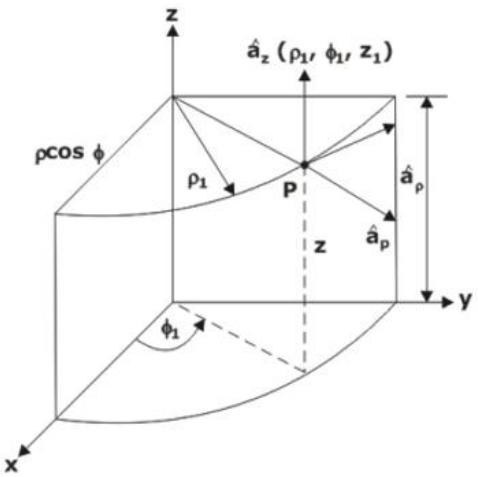



**Figure: Representation of cylindrical coordinates**

Notice that the unit vectors $\hat{a}_\rho, \hat{a}_\phi$ and $\hat{a}_z$ are mutually perpendicular because our coordinates system is orthogonal.

$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_\phi \times \hat{a}_z = \hat{a}_z \times \hat{a}_\rho = 0$

$\hat{a}_\rho \times \hat{a}_\rho = \hat{a}_\phi \times \hat{a}_\phi = \hat{a}_z \times \hat{a}_z = 1$

$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$

$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$

**Conversion of cartesian coordinate to cylindrical coordinate and vice-versa**

Point transformation,

$\rho = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1} \dfrac{y}{x}$

$z = z$

or

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

The relationship between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ are vector transformation,

$\hat{a}_x = \cos \phi \, \hat{a}_\rho - \sin \phi \, \hat{a}_\phi$

$\hat{a}_y = \sin \phi \, \hat{a}_\rho + \cos \phi \, \hat{a}_\phi$

$\hat{a}_z = \hat{a}_z$

or $\hat{a}_\rho = \cos\phi\hat{a}_x + \sin\phi\hat{a}_y$

$\hat{a}_\phi = -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y$

$\hat{a}_z = \hat{a}_z$

Finally, the relationship between ($A_x$, $A_y$, $A_z$) and $\left(A_\rho, A_\phi, A_z\right)$ are

$$\begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix}$$

### 3.3. Spherical Coordinates

A point P can be represented as ($\left(r, \theta, \phi\right)$) and s illustrated in figure below. From figure, we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the colatitudes) is the angle between the z-axis and the position vector of P; and $\phi$ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$0 \le r \le \infty$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

**Note:** the unit vectors $\hat{a}_r, \hat{a}_\theta$ and $\hat{a}_\phi$ are mutually perpendicular because our coordinate system is orthogonal.

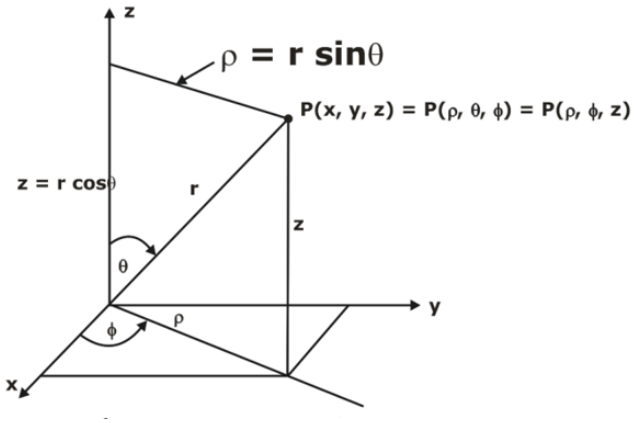
$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_\phi \times \hat{a}_r = 0$

$\hat{a}_r \times \hat{a}_r = \hat{a}_\theta \times \hat{a}_\theta = \hat{a}_\phi \times \hat{a}_\phi = 1$

$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$

$\hat{a}\theta \times \hat{a}_\phi = \hat{a}_r$

$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$




**Conversion of cartesian coordinate to spherical coordinate and vice-versa**

Point transformation,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2 + z^2}}{z}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

Or

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

The relationship between $\hat{a}_x, \hat{a}_y \hat{a}_z$ and $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ are

$$\hat{a}_x = \sin\theta \cos\phi \, \hat{a}_r + \cos\theta \cos\phi \, \hat{a}_\theta - \sin\phi \hat{a}_\phi$$

$$\hat{a}_y = \sin\theta \sin\phi \, \hat{a}_r + \cos\theta \sin\phi \, \hat{a}_\theta + \cos\phi \hat{a}_\phi$$

$$\hat{a}_z = \cos\phi \hat{a}_r - \sin\phi \hat{a}_\phi$$

$$\hat{a}_r = \sin\theta \cos\phi \hat{a}_x + \sin\theta \sin\phi \hat{a}_y + \cos\phi \hat{a}_z$$

$$\hat{a}_\theta = \cos\theta \cos\phi \hat{a}_x + \cos\theta \sin\phi \hat{a}_y - \sin\phi \hat{a}_z$$

$$\hat{a}_\phi = -\sin\phi \hat{a}_x + \cos\phi \hat{a}_y$$

Finally, the relationship between ($A_x$, $A_y$, $A_z$) and $\left(A_r, A_\theta, A_\phi\right)$ are

Vector transformation,

$$\begin{vmatrix} A_r \\ A_\theta \\ A\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

$$\begin{vmatrix} A_x \\ A_y \\ Az \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\cos\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\phi & -\sin\phi & 0 \end{vmatrix} \begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix}$$

## 4. DIFFERENTIAL ELEMENTS IN COORDINATE SYSTEMS

### 4.1. Differential Elements in Rectangular Coordinate System

The differential elements in rectangular coordinate system are defined as follows:

1. Differential length in rectangular coordinate system:

   $dL = dx\alpha_x + dy\alpha_y + dz\alpha_z$

2. Differential area in rectangular coordinate system:

   $dS = dydz\alpha_x = dxdz\alpha_y = dxdy\alpha_z$

3. Differential volume in rectangular coordinate system:

   $dV = dxdydz$

### 4.2. Differential Elements in Cylindrical Coordinate System

The differential elements in cylindrical coordinate system are defined as follows:

1. Differential length in cylindrical coordinate system:

   $dL = d\rho\alpha_\rho + \rho d\varphi\alpha_\varphi + dz\alpha_z$

2. Differential area in cylindrical coordinate system:

   $dS = \rho d\varphi dz\alpha_\rho = d\rho dz\alpha_\varphi = \rho d\varphi d\rho\alpha_z$

3. Differential volume in cylindrical coordinate system:

   $dV = \rho d\rho d\varphi dz$

### 4.3. Differential Elements in spherical Coordinate System

The differential elements in spherical coordinate system are defined as follows:

1. Differential length in spherical coordinate system

   $dL = dr\alpha_r + rd\theta\alpha_\theta + r\sin\theta d\varphi\alpha_\varphi$

2. Differential area in spherical coordinate system

   $dS = r^2\sin\theta d\theta d\varphi\alpha_r = r\sin\theta dr d\varphi\alpha_\theta = rdrd\varphi\alpha_\varphi$

3. Differential volume in spherical coordinate system

   $dV = r^2\sin\theta dr d\theta d\varphi$

## 5. DIFFERENTIAL CALCULUS

The Del operator ($\nabla$), in the different coordinate system, is defined as

$$\nabla = \frac{\partial}{\partial x}\alpha_x + \frac{\partial}{\partial y}\alpha_y + \frac{\partial}{\partial z}\alpha_z \qquad \text{(Rectangular coordinates)}$$

$$\nabla = \frac{\partial}{\partial \rho}\alpha_\rho + \frac{1\partial}{\rho\partial\phi}\alpha_\phi + \frac{\partial}{\partial_z}\alpha_z \qquad \text{(Cylindrical Coordinates)}$$

$$\nabla = \frac{\partial}{\partial r}\alpha_r + \frac{1}{r}\frac{\partial}{\partial\theta}\alpha_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\alpha_\phi \qquad \text{(Spherical coordinates)}$$

## 6. Gradient of a Scalar

The gradient (or grad) is defined by the operation of the Del operator on a scalar field. For a scalar Field V, we define the gradient in the different coordinates as

$$\nabla V = \frac{\partial V}{\partial x}\alpha_x + \frac{\partial V}{\partial y}\alpha_y + \frac{\partial V}{\partial z}\alpha_z \qquad \text{(Rectangular coordinates)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\alpha_\rho + \frac{1\partial V}{\rho \partial \phi}\alpha_\phi + \frac{\partial V}{\partial z}\alpha_z \qquad \text{(Cylindrical coordinates)}$$

$$\nabla V = \frac{\partial V}{\partial r}\alpha_r + \frac{1\partial V}{r\partial \theta}\alpha_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\alpha_\phi \qquad \text{(Spherical coordinates)}$$

## 7. Divergence of a Vector

Divergence of a vector function is a scalar and defined as the net outward flux per unit volume over the elementary closed surface. For a vector function A, we define the divergence in the different coordinates as

$$\nabla \cdot A = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \qquad \text{(Rectangular)}$$

$$\nabla \cdot A = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad \text{(Cylindrical)}$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad \text{(Spherical)}$$

## 8. Curl of a Vector

The curl of a vector plays a very important role in electromagnetic theory.

We define the curl of vector A in different coordinate systems as

$$\nabla \times A = \begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix} \qquad \left(\text{Rectangular coordinates}\right)$$

$$\nabla \times A = \left(\frac{1}{\rho}\right)\begin{bmatrix} a_x & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A\phi & A_z \end{bmatrix} \qquad \left(\text{Cylindrical coordinates}\right)$$

$$\nabla \times A = \left(\frac{1}{r^2\sin\theta}\right)\begin{vmatrix} a_\rho & r a_\theta & r\sin\theta a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \qquad \text{(Spherical coordinates)}$$

## 9. Laplacian Operator

The Laplacian Operator is the square of the Del operator and written as ($\nabla^2$). It can operate both on scalar as well as vector field. The Laplacian of a scalar field is a scalar field whereas the Laplacian of a vector is a vector field.

### 9.1. Laplacian of a Scalar

The Laplacian of a scalar field V in different coordinate systems is defined as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$ (Rectangular coordinates)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$ (Cylindrical coordinates)

$$\nabla^2 . V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ (Spherical coordinates)

### 9.2. Laplacian of a Vector

The Laplacian of a vector is defined as the gradient of divergence of the vector minus the curl of the curl of vector, i.e.

$\nabla^2 A = \nabla (\nabla . A) - \nabla \times \nabla \times A$

## 10. DIVERGENCE THEOREM

According to divergence theorem, the surface integral of a vector field over a closed surface is equal to the volume integral of the divergence of the vector field over the volume. Mathematically, the divergence theorem is written as

$$\oint_s A \cdot dS = \int_v (\nabla \cdot A) dV$$

Where A is a vector field and V is the volume bounded by the closed surface S.

## 11. STOKE'S THEOREM

According to Stoke's theorem, the line integral of a vector field around a closed path in equal to the surface integral of the curl of vector field over the open surface bounded by the closed path. Mathematically, the Stoke's theorem is written as

$$\oint_L A \cdot dL = \int_S (\nabla \times A) \cdot dS$$

Where A is a vector field and S is the open surface bounded by the closed path L.

# CHAPTER 2: ELECTROSTATICS

## 1. Electric charge

Electric Charge is a fundamental conserved property of some subatomic particles, which determines their electromagnetic interaction.

### 1.1 Point Charge

Point charges are very small charges assumed to be of infinitesimally small volume, although they have finite volume considered as a single charge.

### 1.2. Line Charge

The charge per unit length along the line charge is called line charge density. It is denoted by $\rho_L$ and defined as

$$\rho_L = \lim_{\Delta L \to 0} \frac{\Delta Q}{\Delta L} = \frac{dQ}{dL}$$

where $\Delta Q$ is small charge, and $\Delta L$ is small length.

### 1.3. Surface Charge

The charge per unit area over the surface is called the surface charge density. It is denoted by $\rho_S$ and defined as

$$\rho_S = \lim_{\Delta s \to 0} \frac{\Delta Q}{\Delta S} = \frac{dQ}{dS}$$

where $\Delta Q$ is small charge, and $\Delta S$ is small area.

### 1.4. Volume Charge

The charge per unit volume in the region is called volume charge density. It is denoted by $\rho_v$ and defined as

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v} = \frac{dQ}{dv}$$

where $\Delta Q$ is small charge, and $\Delta v$ is small volume.

## 2. Electric flux Density

The electric flux density vector $\vec{D}$ in a medium is defined as the product of the permittivity and the electric field vector

$$\vec{D} = \in E$$

The permittivity of the medium is defined in terms of the free space permittivity and the relative permittivity $(\in')$as

$$\in = \in' \in_0$$

Electric flux density is independent of the medium properties

For point charge $\vec{E} = \frac{Q}{4\pi \in R^2} \hat{a}_R, \vec{D} = \frac{Q}{4\pi R^2} \hat{a}_R$

For line charge $\vec{E} = \frac{P_L}{2\pi \in_P} \hat{a}_P, \vec{D} = \frac{P_L}{2\pi_P} \hat{a}_P$

The unit of electric flux density are

$$\frac{F}{m} \times \frac{F}{m} = \frac{C}{m^2}$$

**NOTE:** The units of $\vec{D}$ are equivalent to surface density i.e. $C/m^2$

**3. Gauss's Law – Maxwell Equations**

The total outward electric flux Ψ through any closed surface is equal to the total charge enclosed by the surface.

In equation form, gauss's law is written as

$$\psi = \oint_s \vec{D}.\, d\vec{S} = Q_{enclosed}$$

Where $= d\vec{S} = \hat{a}_n dS$ and $\hat{a}_n$ is the outward pointing unit normal to closed surface S.

$\psi = \oint_s \vec{D}.\, d\vec{S} =$ total charge enclosed

$$Q = \oint_v P_v dv$$

or $Q = \oint_S \vec{D}.d\vec{S} = \int_V \rho_v dv$

By applying divergence theorem to the middle term, we have

$$\oint_S \vec{D}.d\vec{S} = \int_V \nabla.\vec{D} dv$$

Comparing the two volume integrals

$$\nabla.\vec{D} = \rho_v$$

It states that the volume charge density is the same as the divergence of the electric flux density.

**4. Electric field due to a point charge**

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_\tau$$

$$\vec{E} = \frac{Q}{4\pi \in_0 r^2}\hat{a}_\tau$$

Where, Q is the point charge and r is the distance between point where electric field is calculated and point charge.

**5. Electric field due to an Infinite line charge**

$$\Rightarrow \vec{D} = \frac{P_L}{2\pi\rho}\hat{a}_P$$

$$and \quad \vec{E} = \frac{P_L}{2\pi \in_0 \rho}\hat{a}_P$$

Where, $P_L$ is linear charge density, $\rho$ is distance of the point P (P is the point where electric field is calculated) from line charge and $\hat{a}_P$ is position vector of point P.

**6. Electric Field due to an infinite sheet of charge**

$$\vec{D} = \frac{\rho_s}{2}\hat{a}_z$$

Or, $\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0}\hat{a}_z$

Where, $\rho_s$ is surface charge density and $\hat{a}_z$ is the unit normal vector from sheet to the pointwhere electric field is calculated.

**7. Field due to a uniformly chargedsphere**

$\vec{D} = \begin{cases} \frac{r}{3}\rho_v \hat{a}_\tau; & 0 < r \le a \\ \frac{a^3}{3r^2}\rho_v \hat{a}_\tau; & r \ge a \end{cases}$ Where, $\rho_v$is volume charge density.

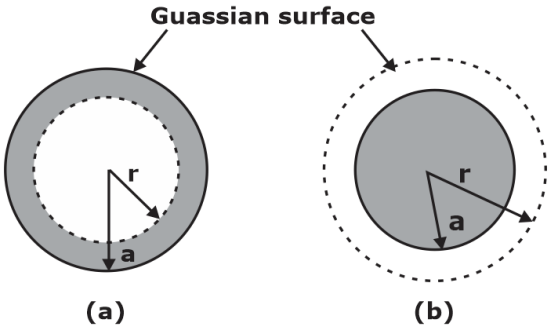



**Figure:** Gaussian surface for a uniformly charged when (a)r ≥ a and (b) r ≤ a

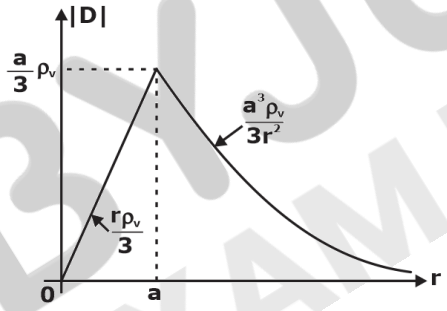



**Figure:** Sketch of |D| against r for a uniformly charged sphere.

**8. Electric field due to multiple point charger**

The electric field due to multiple points chargers can be determined using the principle of superposition. for N point charges $Q_1, Q_2, \ldots\ldots Q_N$ located at $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots\ldots \vec{r}_N$ the electric fields intensity at point $\vec{r}$ is obtained by equations.

$$\vec{E} = \frac{Q_1(\vec{r} - \vec{r}_1)}{4\pi\epsilon_0 \left|\vec{r} - \vec{r}_1\right|^3} + \frac{Q_2(\vec{r} - \vec{r}_2)}{4\pi\epsilon_0 \left|\vec{r} - \vec{r}_1\right|^3} \ldots + \frac{Q_N(\vec{r} - \vec{r}_N)}{4\pi\epsilon_0 \left|\vec{r} - \vec{r}_N\right|^3}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{K=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_k)}{\left|\vec{r}_1 - \vec{r}_k\right|^3}$$

**9. Electric field due to charge distributions**

$$dQ = \rho_L dl \Rightarrow Q = \int_L \rho_L dl (\text{line charge})$$

$$dQ = \rho_S dS \Rightarrow Q = \int_S \rho_S dS \text{ (surface charge)}$$

$$dQ = \rho_v dV \Rightarrow Q = \int_v \rho_v dV \text{ (volume charge)}$$

**10. Electric field on the axis of a charged ring**

Consider a circular ring of radius a with uniform line charge density $\rho_L$(C/m) and a point P on the axis of ring as shown in figure

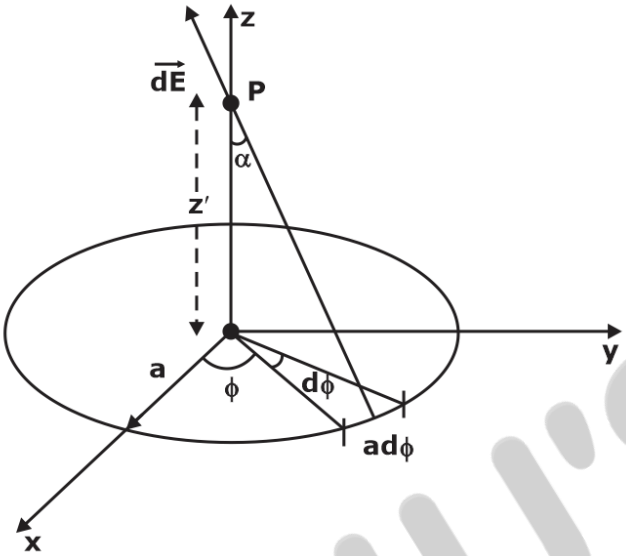



**Figure:** electric field to circular ring

The total electric field is therefore

$$\vec{E} = \frac{Qz}{4\pi\varepsilon_0(z^2 + a^2)^{3/2}}\hat{a}_z$$

Note: $\text{As } z \to \infty, \vec{E} \text{ tends to } \frac{Q}{4\pi\varepsilon_0 z^2}$

**11. Electric field of a Charged Circular Disk**

The electric field due to a uniformly charged circular disk at a point on its axis can be calculated using the result for a ring. Consider a disk of radius a, surface charge density $\rho_s$*(C/m$^2$) and point P as shown in the figure

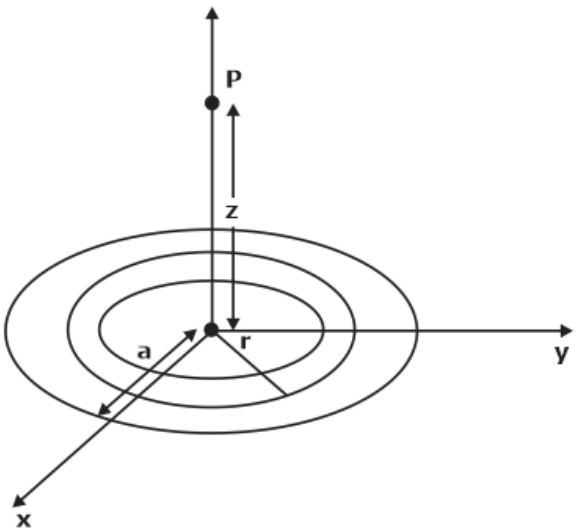



**Figure:** Electric field due to charged circular

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)\hat{a}_z$$

Note: If $z \gg a$ then $\vec{E} = \frac{Q}{4\pi \in_0 z^2}\hat{a}_z$

**12. Potential difference in the electric field of a point charge**

The potential difference between two points *A* and *B* in the electric fields of the point charge is

$$V_{AB} = -\int_A^B \vec{E}.d\vec{l} = \int_A^B dV = V_B - V_A$$

**13. Energy Density in Electrostatic Field**

$$W_E = \frac{1}{2}\int_{vol.} (D.E)\, dv = \frac{1}{2}\int_{vol.} \in_0 E^2 dv \text{ (J)}$$

Where, D is electric flux density and E is electric field intensity.

We defined energy density in ($J/m^3$)

**14. Boundary Conditions**

Electric field intensity $\vec{E}$ into two orthogonal components

$$\vec{E} = \vec{E}_t + \vec{E}_n$$

where $\vec{E}_t$ and $\vec{E}_n$ are tangential and normal components of $\vec{E}$ respectively.

**14.1. Dielectric-Dielectric Boundary Conditions**

Consider the $\vec{E}$ field existing in a region that consist of two different dielectrics characterized by $\epsilon_1 = \epsilon_0\epsilon_n$ and $\epsilon_2 = \epsilon_0\epsilon_a$ as shown in figure.

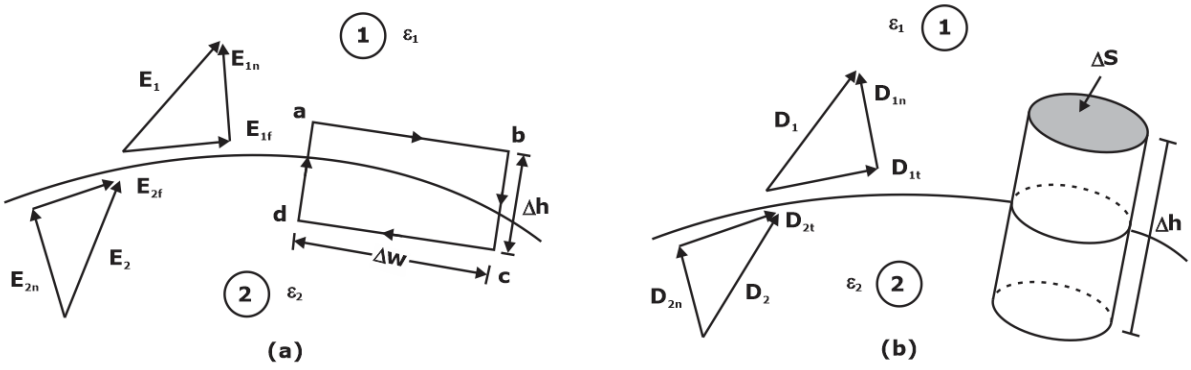



**Figure: Dielectric-dielectric boundary: (a) determining $E_{1t} = E_{2t}$ (b) determining $D_{1\sigma} = D_{2\sigma}$.**

The fields and $\vec{E}_1$ and $\vec{E}_2$ can be decomposed as

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

then, $E_{1t} = E_{2t}$

### 14.2. Conductor-Dielectric Boundary Conditions

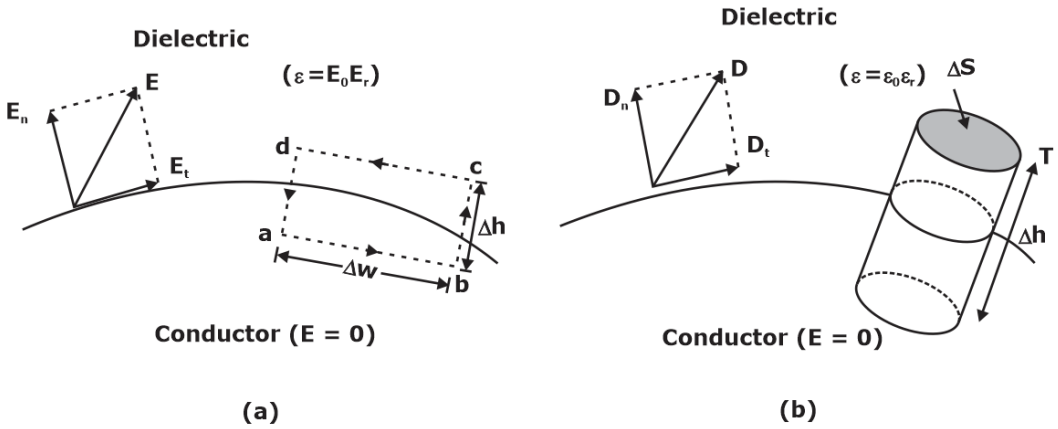



**Figure: Conductor-dielectric boundary.**

$$D_n = \frac{\Delta Q}{\Delta S} = \rho_S$$

or, $D_n = \rho_S$

### 14.3. Conductor-Free Space Boundary Conditions

The boundary conditions at the interface of conductor and free space can be obtained from conductor-dielectric boundary conditions with $\epsilon_r = 1$.

Thus the boundary condition are

$E_t = 0$. $D_t = \epsilon_0 E_t = 0$

$$D_n = \frac{\rho_S}{\epsilon_0}$$

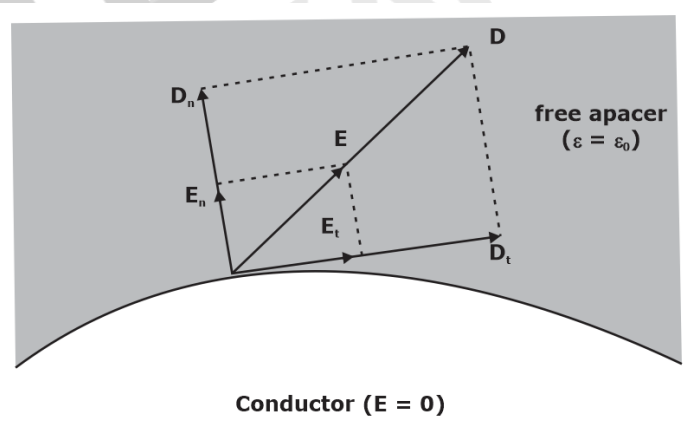



**Figure: Conductor-free space boundary**

## 15. Poisson's and Laplace's Equations

$\nabla^2 V = -\frac{\rho_v}{\epsilon}$ , Where V is electrostatic potential and $\rho_v$ is volume charge density.

This is known as Poisson's equation.

As special case of this equation occurs when $\rho_v = 0$ (i.e., for a charge free region

$$\nabla^2 V = 0$$

Which is known as Laplace's equation.

## 16. Coaxial Capacitor

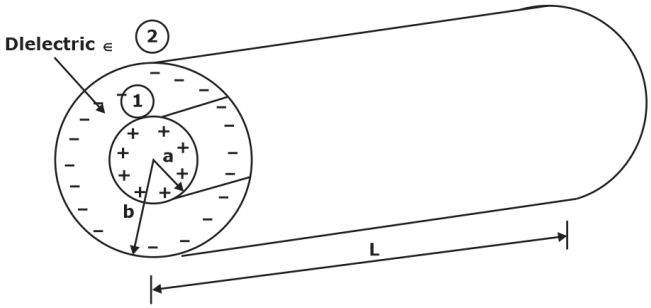



**Figure:** A coaxial capacitor

The capacitance of a coaxial cylinder is given by

$$C = \frac{Q}{V} = \frac{2\pi \in}{\ln \frac{b}{a}}$$

## 17. Spherical Capacitor

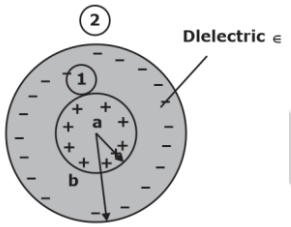



**Figure:** A spherical capacitor

The capacitance of the spherical capacitor is

$$C = \frac{Q}{V} = \frac{4\pi \in}{\frac{1}{a} - \frac{1}{b}}$$

# CHAPTER 3: MAGNETOSTATICS

## 1. MAGNETIC FLUX DENSITY

Magnetic flux density is the amount of magnetic flux per unit area of a section, perpendicular to the direction of magnetic flux.

It is denoted by B. Mathematically,

$$B = \frac{d\Phi}{dS} a_n$$

Where $d\Phi$ is a small amount of magnetic flux through small area dS of the section perpendicular to magnetic flux $a_n$ is the unit vector normal to the surface area.

also expressed as

$$\Phi = \int_S B \bullet dS$$

## 2. Relation between Magnetic field Intensity (H) and Magnetic Flux Density (B):

The magnetic field intensity is related to the magnetic flux density as

$$B = \mu H = \mu_0 \mu, H$$

Where, $\mu$ is the permeability of the medium, $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of free space, and $\mu$, is the relative permeability of the medium.

## 3. Biot-Savart's Law

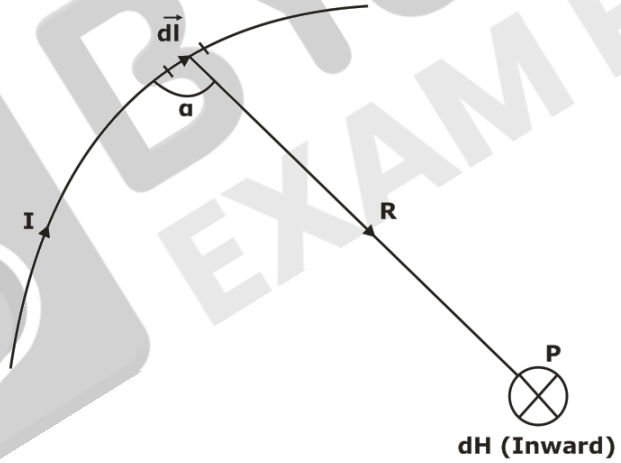



**Figure: Magnetic field due to small length at P**

Line current $\vec{H} = \int_L \frac{I\vec{dl} \times \vec{a}_R}{4\pi R^2}$

Surface current $\vec{H} = \int_L \frac{\vec{K}dS \times \vec{a}_R}{4\pi R^2}$

Volume current $\vec{H} = \int_V \frac{\vec{J}dv \times \vec{a}_R}{4\pi R^2}$

(H direction = I direction × Radial vector)

**4. Ampere's Circuital Law**

According to Ampere's circuital law the line integral of magnetic field intensity H around the closed loop L is equal to I, i.e.

$$\oint_L H \cdot dL = I$$

**Differential Form of Ampere's Circuital Law**

In differential form Ampere's circuital law is defined as

$\nabla \times H = J$

i.e. the curl of the magnetic field intensity (H) is equal to the current density (J) at the point in space.

**5. H-field for finite length of current I carrying wire:**

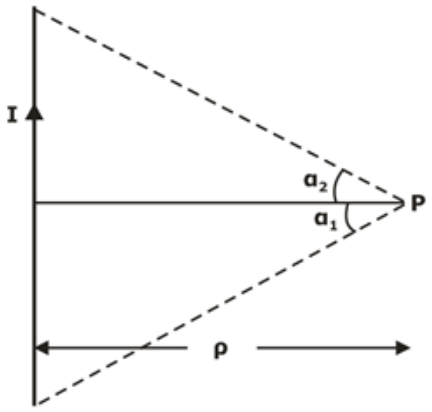



**Figure: Field at P due to line conductor**

$$\vec{H} = \frac{I}{4\pi\rho}(\sin\alpha_1 + \sin\alpha_2)\,a_\phi$$

**Note:** Notice from the above equation that $\vec{H}$ is always along the unit vector $\hat{a}_\phi$ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P.

**6. H-field for infinite length of current I carrying wire:**

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

The unit vector $\hat{a}_\phi$ must be found carefully. A simple approach is to determine $\hat{a}_\phi$ form

$\hat{a}_\phi = \hat{a}_l \times \hat{a}_p$

Where $\hat{a}_l$ is a unit vector along the line current and $\hat{a}_p$ is a unit vector along the perpendicular line from the line current to the filed point.

| | **Electric force ($F_e$ = QE)** | **Magnetic Force ($F_m$ = Qu ×B)** |
|---|---|---|
| 1. | It is in the same direction as the field E. | It is perpendicular to both u and B. |
| 2. | It can perform work. | It cannot perform work. |

| | | |
|---|---|---|
| 3. | It is independent of the velocity of charge. | It depends upon the velocity of charge. |
| 4. | It can produce change in kinetic energy. | It cannot produce change in kinetic energy. |

**Table: Comparison between Electric Force and Magnetic Force**

**7. Force on a Differential Current Element in Magnetic Field**

The differential magnetic force experienced by various differential current elements are given below:

$F_m = \int_L IdL \times B$ (Line current)

$F_m = \int_S K \times BdS$ (Surface current)

$F_m = \int_V J \times Bdv$ (Volume current)

Where IdL is the line current element, KdS is surface current element, Jdv is volume current element, and $F_m$ is the magnetic force exerted on the respective elements in presence of magnetic field B

**8. Magnetic Force Between Two Current Elements**

Consider the two differential current elements $I_1dL_1$ and $I_2dL_2$ separated by a distance r. The magnetic force between the two current elements is given by

$$F = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} \frac{dL_2 \times (dL_1 \times a_r)}{r^2}$$

This equation is also called Ampere's force law.

**9. Magnetic Susceptibility**

In a linear material, magnetization is directly proportional to field intensity. i.e.

$M \propto H$

or $M = \chi_m H$

where $\chi_m$ is the magnetic susceptibility of the medium. The magnetic susceptibility of a magnetic material is a measure of the degree of magnetization of a material in response to an applied magnetic field.

**10. Relation between Magnetic Field Intensity and Magnetic Flux Density**

In a magnetic material, magnetic flux density is expressed in terms of magnetic field intensity as

$B = \mu_0(H + M) = \mu_0(1 + \chi_m)H$
$= \mu_0\mu_r H = \mu H$

where

$\mu = \mu_0\mu_r$ is called permeability of the medium, expressed in Henry per metre (H/m),

$\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of free space, known as absolute permeability,

$\mu_r = (1 + \chi_m) = \frac{\mu}{\mu_0}$ is the relative permeability of the medium, it is dimensionless.

**11. Energy Density in a Magnetic Field**

In a magnetic field with flux density B, the stored magnetic energy density is given by

$$w_m = \frac{1}{2}(B \cdot H)$$

where H is the magnetic field intensity in the region. The total magnetic energy stored in a region is obtained by taking the volume integral of the energy density, i.e.

$$W_m = \int_v w_m dv = \int_v \frac{1}{2}(B \cdot H)\, dv$$

**12. Boundary Conditions for Magnetostatic Fields:**

$B_{1n} = B_{2n}$ States that Normal component of B is continuous across an interface. $\mu_1 H_{1n} = \mu_2 H_{2n}$ $H_{1t} - H_{2t} = J_{sn}$ States that the Tangential component of H field is discontinuous across an interface where free surface current exist-amount the amount of discontinuity being equal to the surface current density.

When conductivities of both media are finite, current are defined by volume current densities and free surface currents don't exist on interface hence j equal to zero, and the Tangential component of H field is continuous across the boundary of almost all physical media; it is discontinuous only when an interface with an ideal conductor or a super conductor is assumed.

## CHAPTER 4: MAXWELL'S EQUATION

**Maxwell Equations**

| Differential form | integral form | Significance |
|---|---|---|
| $\nabla \times E = -\frac{\partial B}{\partial t}$ | $\oint E.dl = -\iint \frac{\partial B}{\partial t}.ds$ | Faraday's Law |
| $\nabla \times H = J + \frac{\partial D}{\partial t}$ | $\oint H.dl = -\iint (J + \frac{\partial D}{\partial t}).ds$ | Ampere's Circuital Law |
| $\nabla.D = \rho_v$ | $\oiint D.ds = Qenclosed$ | Gauss Law |
| $\nabla.B = 0$ | $\oiint B.ds = 0$ | No isolated magnetic charge |

****

BYJU'S EXAM PREP

# Electrical Engineering

## Electrical Machines

SHORT NOTES

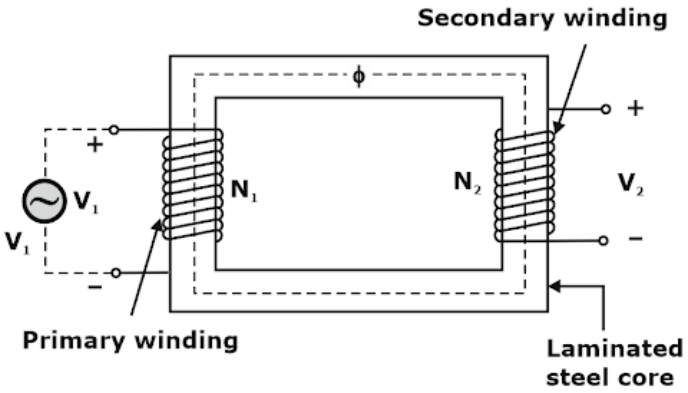
# ELECTRICAL MACHINES (FORMULA NOTES)

## Transformer

### Features of Single-phase Transformers

Transformer is a static device which performs:

- It works on the principle of electromagnetic induction.
- Transfers electrical energy from one electrical circuit to another without changing of frequency.
- The energy transfer usually occurs with voltage change, although this is not always necessary.




**Basic Diagram of Transformer**

- When the transformer raises the voltage, i.e. when output voltage of a transformer is higher than its input voltage, it is called step-up transformer and when output voltage is lowered, it is called step-down transformer.
- Transformer doesn't change the frequency of the system. Hence, it can be treated as **constant frequency device**, it transfers almost same amount of power from one circuit to another so it can be treated as **constant power device** and the amount of flux in its core remains constant, so it is also known as **constant flux device**.

### Ideal Transformer

- Ideal transformer does not have any loss, so resistances are also assumed to be zero.
- Magnetic leakage flux in transformer is completely zero. (K=1, coefficient of coupling)
- Magnetic permeability of transformer core $\mu = \infty$. So, the flux can be established without any excitation current.
- Magnetization curve of transformer core is assumed to be linear.
- For an ideal two-winding transformer with primary voltage $V_1$ applied across $N_1$ primary turns and secondary voltage $V_2$ appearing across $N_2$ secondary turns:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

- The primary current $I_1$ and secondary current $I_2$ are related by:

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} = \frac{V_2}{V_1}$$

- For a single-phase transformer with rated primary voltage $V_1$, rated primary current $I_1$, rated secondary voltage $V_2$ and rated secondary current $I_2$, the volt ampere rating S is:

$$S = V_1 I_1 = V_2 I_2$$

- During operation of transformer:

$$B_m \propto \frac{E_1}{f} \propto \frac{V_1}{f}$$

- $B_{max}$ = Constant $\Rightarrow \frac{V_1}{f} = \text{Constant}$

**Note:**

- Gross cross-sectional area = Area occupied by magnetic material + Insulation material.
- Net cross-sectional area = Area occupied by only magnetic material excluding area of insulation material.
- Hence, net cross-sectional area is taken for all calculations since majority $\phi(\text{flux})$ flows in magnetic material.

  $\phi = BA_n$

## E.M.F. EQUATION OF TRANSFORMER

Let, Flux $\Phi = \Phi_m \sin(\omega t)$

As the flux is assumed to be sinusoidal,

**In primary winding side:**

According to faraday's law,

$$e_1 = -N_1 \frac{d\phi}{dt} = -N_1 \frac{d}{dt}(\phi_m \sin \omega t)$$

Instantaneous value of emf in primary

$$e_1 = N_1 \phi_m \omega \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$\boxed{E_{1rms} = \sqrt{2}\pi f \phi_m N_1}$$

**In secondary winding side**

Similarly, $\boxed{E_{2rms} = \sqrt{2}\pi f \phi_m N_2}$

$$\frac{E_1}{E_2} = \frac{N_1}{N_2}$$

Also, $\frac{E_1}{N_1} = \frac{E_2}{N_2} = \sqrt{2}\,\pi\, f\, \phi_m$

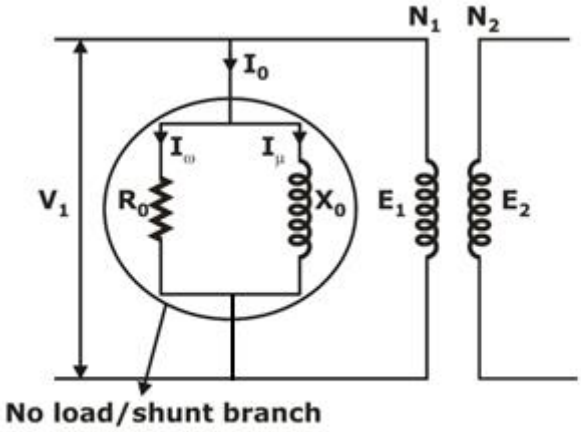
From the above equation,

EMF per turn in primary = EMF per turn in secondary

**Note:**

- Transformation ratio = $K = \frac{E_2}{E_1} = \frac{N_2}{N_1}$
- Turn ratio = $\frac{1}{K} = N_1 : N_2$

**Equivalent circuit of Transformer under No-Load condition:**




- No load current/ exciting current = $I_0 = \bar{I}_\mu + \bar{I}_w = I_0 \angle -\phi_0$

  $I_w = I_0 \cos\phi_0$

  $I_\mu = I_0 \sin\phi_0$

  $I_w$ = core loss component

  $I_\mu$ = magnetizing component

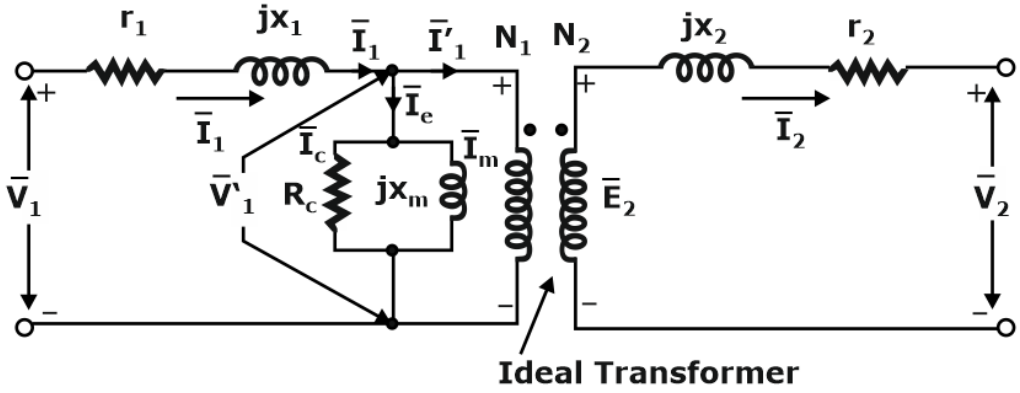
- No load power = $V_1 I_0 \cos\phi_0 = V_1 I_w$ = Iron losses

  $R_0 = \frac{V_1}{I_w}; X_0 = \frac{V_1}{I_\mu} \Rightarrow \quad I_w = \frac{\text{No load power}}{V_1}$

## EQUIVALENT CIRCUIT OF A TRANSFORMER

### Equivalent circuit

The equivalent circuit of a transformer having voltages transformation ratio, $k = \frac{E_2}{E_1}$ is shown in figure below:



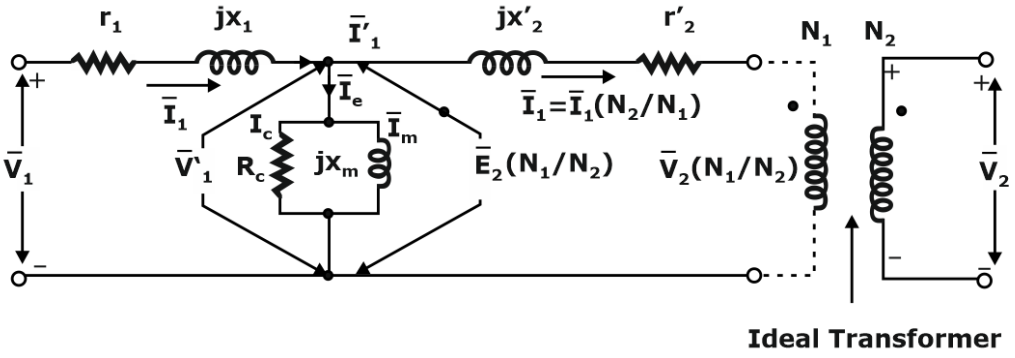

**Exact Equivalent Circuit**

Practical representation of simple two-winding transformer:

$$\vec{E}_2 = \vec{V}_2 + \vec{I}_2\vec{R}_2 + j\vec{I}_2X_2$$

$$\vec{V}_1 = \vec{E}_1 + \vec{I}_1R_1 + j\vec{I}_1X_1$$

**When all the parameters are referred to primary side**

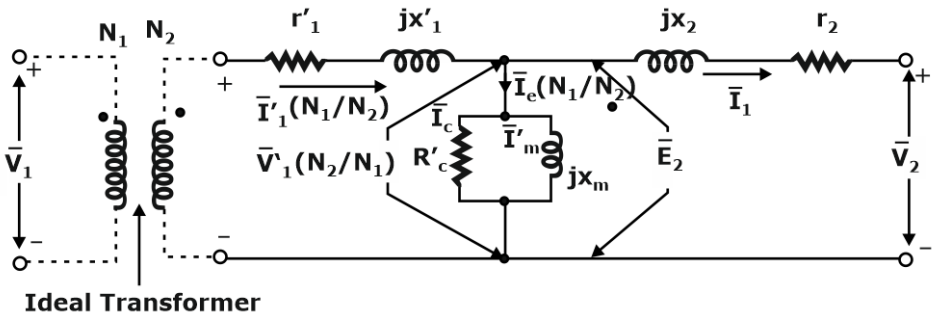



**Referred to Primary side**

$r_{e_1}$ = Equivalent resistance refers to primary $= \left(r_1 + \frac{r_2}{k^2}\right)$

$x_{e_1}$ = Equivalent reactance refers to primary $= \left(x_1 + \frac{x_2}{k^2}\right)$

**When all the parameters are referred to secondary side**




**Referred to Secondary side**

$r_{e_2}$ = Equivalent resistance to secondary side $= \left(r_2 + k^2 r_1\right)$

$x_{e_2}$ = Equivalent reactance refers to secondary side $= \left(x_2 + k^2 x_1\right)$

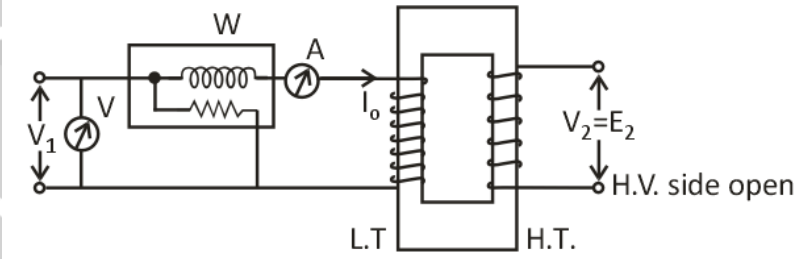
$R_c$ = Core loss resistance

$X_m$ = Magnetising reactance

Where, k = Transformation Ratio $= \frac{N_2}{N_1}$

**TESTS ON TRANSFORMER:**

1. **Open Circuit Test (O.C.) or No-load Test:**
   - This test is performed to determine core or iron loss $P_i$ and no-load current $I_o$.
   - This test is helpful in determining the magnetizing current $I_\mu$ and the core loss components $I_w$ and hence no-load shunt branch parameters $R_o$ and $X_o$.




**Open Circuit Configuration**

Iron loss ($P_i$) = Input no-load Power = $W_o$ (wattmeter reading)

No-load current = $I_o$

Hence no load p.f., $\cos\phi_o = \frac{W_o}{V_1 I_o}$

Now from no-load phasor diagram

$I_w = I_o \cos\phi_o$

$I_\mu = I_o \sin\phi_o$

Now,

$R_o = \frac{V_1}{I_w} = \frac{V_1}{I_o \cos\phi_o}$

$X_o = \frac{V_1}{I_\mu} = \frac{V_1}{I_o \sin\phi_o}$

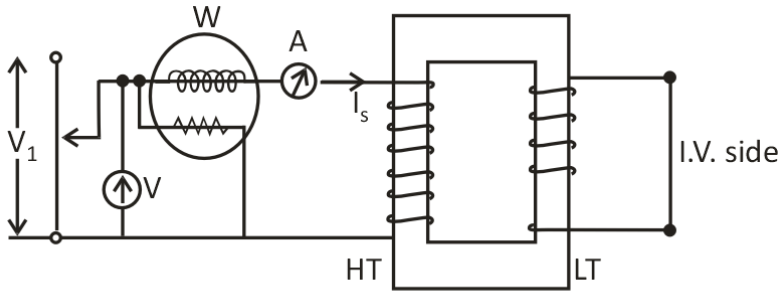
The transformer's output power is zero at no load efficiency.

No load test conducted on rated voltage and rated frequency on LV side of transformer.

**2. Short Circuit Test (S.C.):**

S.C. test is carried out at rated current to determine the Cu loss at full load condition.

Full load Cu-loss, $P_C = I^2_s r_{e_1} = W$




**Short circuit of transformer**

Equivalent resistance referred to HV side, $r_{e_1} = \left(\frac{W}{I_s^2}\right)$

Equivalent impedance refers to HV side, $Z_{e_1} = \left(\frac{V_s}{I_s}\right)$

Thus, equivalent reactance, $X_{e_1} = \sqrt{Z_{e_1}^{\ 2} - r_{e_1}^{\ 2}}$

**Note:** As the applied voltage required to circulate full load current at short-circuit would be limited to 5% to 10% of the rated voltage, the core loss at such reduced voltage is ignored. Also, the exciting current at such low voltage may be neglected.

## VOLTAGE REGULATION OF A TRANSFORMER

It is defined as the percentage change in secondary terminal voltage from no load to full load, expressed as a percentage (or pu) of the secondary rated voltage.

If ,

$V_2$ = secondary terminal voltage at rated load, and

$E_2$ = secondary terminal voltage at no load,

Then,

Voltage regulation $= \frac{V_{NL} - V_{FL}}{V_{FL}}$ (in pu)

Voltage regulation $= \frac{V_{NL} - V_{FL}}{V_{FL}} \times 100$ (in %)

The magnitude of change in secondary voltage depends on the transformer's load power factor, load current, total resistances and total leakage reactance.

Therefore, voltage regulation equation can be given as:

Voltage drop in the secondary terminal voltage,

$$E_2 - V_2 = I_2 r_{e_2} \cos\theta_2 \pm I_2 x_{e_2} \sin\theta_2$$

Hence for any load current $I_2$, Per unit voltage regulation is given by:

$$V.R = \frac{V_{NL} - V_{FL}}{V_{FL}}$$

$$V.R = \frac{E_2 - V_2}{V_2}$$

$$V.R = \frac{I_2 r_{e_2}}{V_2}\cos\theta_2 - \frac{I_2 x_{e_2}}{V_2}\sin\theta_2$$

where,

$\frac{I_2 r_{e_2}}{V_2}$ = PU Equivalent resistance or PU resistance drop = $R_{pu}$

$\frac{I_2 X_{e_2}}{V_2}$ = PU Equivalent reactance or PU reactance drops = $X_{pu}$

At rated load voltage regulation in PU = $\left(R_{pu}\cos\theta_2 \pm X_{pu}\sin\theta_2\right)$

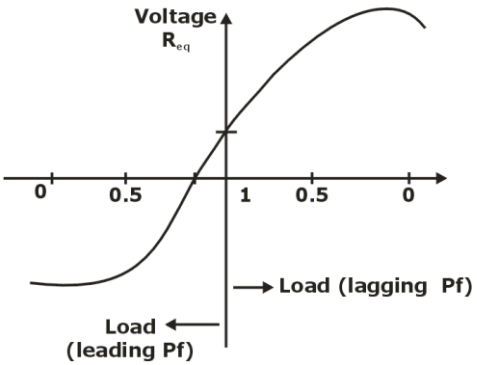
Percentage voltage drop = $\left(R_{pu}\cos\theta_2 \pm X_{pu}\sin\theta_2\right)\times 100$

**Condition for Zero Voltage Regulation:**

Zero voltage regulation takes place at leading pf and when load pf is $\left(\frac{X_{pu}}{Z_{pu}}\right)$.

**Condition for Maximum Voltage Regulation:**

Maximum voltage regulation occurs at lagging power factor and when pf is $\left(\frac{R_{pu}}{Z_{pu}}\right)$




**Plot between voltage regulation and power factor**

- Voltage regulation is a figure of merit of a transformer and its low value is always desired.
- The voltage regulation may be reduced by reducing the per-unit impedance of the transformer.
- Per unit impedance, $Z_{pu}$ can be reduced for reducing $R_{pu}$ or $X_{pu}$.
- Per unit resistance, $R_{pu}$ is directly kept at an optimally low value.
- $X_{pu}$ can be reduced by reducing leakage flux.

**Per Unit System**

A per-unit system is the expression of system quantities as fractions of a defined base unit quantity. Calculations are simplified because quantities expressed as per-unit do not change when they are referred from one side of a transformer to the other.

Per Unit Value = $\frac{\text{Actual value in any unit}}{\text{Base value in same unit}}$

It is usual to assume the base values as:-

- Base voltage = rated voltage of the machine
- Base current = rated current of the machine
- Base impedance = base voltage /base current
- Base power = base voltage x base current

Hence it can be concluded that out of 4 parameters, any two can be selected as base values and according to that the remaining values are determined.

Values which we will choose: 1) Base Apparent Power = $S_B$; 2) Base Voltage = $V_B$

Values which we calculate: 1) Base Current $I_B = \frac{S_B}{V_B}$; 2) Base Impedance $Z_B = \frac{V_B^2}{S_B}$

For an ideal Transformer $E_{1PU} = E_{2PU.}$

Also, $Z_{1PU} = Z_{2PU}$

Direct relation for changing the base of an impedance:

$$Z_{PU_NEW} = Z_{PU_OLD} \times \frac{S_{B_NEW}}{S_{B_OLD}} \times \frac{V_{B_OLD}^2}{V_{B_NEW}^2}$$

**Losses present in transformer**

1. Copper losses: Transformer windings major losses
2. Iron losses: Transformer core
3. Stray load losses: Cu parts and Iron parts have minor losses
4. Dielectric losses: Insulating materials.

**1. Cu losses in transformer:**

Total Cu loss = $I_1^2R_1 + I_2^2R_2 = I_1^2R_{01} = I_2^2R_{02}$

- Rated current on primary side = $\frac{\text{VA rating of transformer}}{E_1}$

- Similarly, current on secondary side = $\frac{\text{VA rating of transformer}}{E_2}$
- Cu losses $\propto I_1^2$ or $I_2^2$. Hence these are called variable losses.
- P.U. full load Cu loss = $\frac{\text{FL Cu loss in watts}}{\text{VA rating of transformer}} = \frac{I_1^2 R_{01}}{E_1 I_1}$
- If VA rating of transformer is taken as base then P.U Cu loss $\propto I_1^2$ as remaining terms are constant.
- P.U. Cu loss at x (Loading factor) of full load = $x^2$ × PU full load Cu loss
- P.U. resistance drop ref to primary $= \frac{I_1 R_{01}}{E_1} \times \frac{I_1}{I_1} = \frac{I_1^2 R_{01}}{E_1 I_1}$

  $\therefore$ P.U. resistance drop = P.U. Full Load Cu loss

**2. Iron (or) Core losses in Transformer:**

**2.1 Hysteresis loss:**

Steinmetz formula:

$W_h = \eta B_{max}^x f V$

Area under one hysteresis loop.

where,

$\eta$ = Stienmetz coefficient

$B_{max}$ = Maximum flux density in transformer core.

f = Frequency of magnetic reversal = supply frequency.

V = Volume of core material

x = Hysteresis coefficient (or) stienmetz exponent = 1.6 (Si or CRGO steel)

**2.2 Eddy current loss:**

Eddy current loss, $W_e \propto R_{ce} \times I_e^2$

As area decreases in laminated core resistance as a result conductivity decrease.

$W_e = K(\text{Constant}) B_{max}^2 f^2 (\text{Supply freq.}) \times t^2 (\text{Thickness of lamination})$

(It is a function of σ)

During operation of transformer:

$B_m \propto \frac{V_1}{f}$

**Case(i):** $\frac{V_1}{f}$ = Constant, $B_{max}$ = Constant

$W_h \propto f \quad , \quad W_e \propto f^2$

$W_h = Af \quad , \quad W_e = Bf^2$

$\therefore \left.\begin{aligned} W_i &= W_h + W_e \\ W_i &= Af + Bf^2 \end{aligned}\right\}$ When $B_{max}$ = Constant

**Case(ii):** $\frac{V_1}{f} \neq$ Constant, $B_{max} \neq$ Constant

$$W_h \alpha \left(\frac{V_1}{f}\right)^x f \qquad , \qquad W_e \propto \left(\frac{V_1}{f}\right)^2 f^2$$

$$W_h \alpha \frac{V_1^x}{f^{x-1}} \qquad W_e \propto V_1^2$$

$W_i = W_h + W_e$

$$W_i = A\frac{V_1^x}{f^{x-1}} + BV_1^2$$

**P.U. iron loss:**

- P.U. iron loss = $\frac{\text{Iron loss in watts}}{\text{VA rating of transformer}}$
- As VA rating is chosen as base, the P.U. iron loss is also constant at all load conditions.

**To find out constant losses:**

- $W_0$ = Losses in transformer under no-load condition = Iron losses + Dielectric loss + no-load primary loss ($I_0^2R_1$)
- Constant losses = $W_0 - I_{02}^2 R_1$
  Where, $R_1$ = LV winding resistance.

**To find out variable losses:**

- $W_{sc}$ = Loss in transformer under SC condition = Full Load Cu loss + stray load losses (Cu and Iron) + Iron losses in both windings.
- Variable losses = $W_{sc}$ – Iron losses corresponding to $V_{SC}$.

## TRANSFORMER EFFICIENCY

### Efficiency Calculation:

$$\eta = \frac{\text{Output power}}{\text{Input power}} = \left(\frac{V_2 I_2 \cos\theta_2}{V_2 I_2 \cos\theta_2 + P_c + I_2^2 r_{e2}}\right)$$

where, $P_i$ = core losses

$I_2^2 r_{e_2}$ = ohmic losses

$V_2 I_2$ = output kVA

cos $\theta_2$ = load p.f.

$$\therefore \qquad \eta = \left(1 - \frac{\text{Losses}}{\text{Input power}}\right)$$

**Note:** Maximum efficiency takes place when variable ohmic losses are equal to the fixed core loss $P_i$.

$$\therefore \quad I_2^2 r_{e_2} = P_i$$

$$\Rightarrow \quad I_2 = \sqrt{\frac{P_i}{r_{e_2}}}$$

$$X = \sqrt{\frac{P_i}{full\ load\ ohmic\ loss}}$$

$$\therefore \quad (kVA)_{\max\ \eta} = (kVA)_{fl}\sqrt{\frac{P_i}{I_{fl}^2 r_{e_2}}}$$

$$S\eta_{max} = V \times I_j \sqrt{\frac{P_I}{I_j^{\,2} R_{eq}}}$$

$$S\eta_{max} = S_j \times \sqrt{\frac{P_I}{P_{cu(j)}}}$$

Here, $S\eta_{max}$ represents the KVA at the maximum efficiency.

$I_j$ – any current at which copper loss is known.

$S_j$ – any kVA at which the Cu-loss is known.

**All Day Efficiency:**

$$\text{All Day efficiency} = \eta_{\text{all-day}} = \frac{\text{Output in KWh}}{\text{Input in KWh}} (\text{for 24 hours})$$

$$\boxed{\eta_{\text{all day}} = \frac{\text{output kWh in 24hrs}}{\text{input kWh in 24hrs}}}$$

$$\boxed{\frac{\text{output KWh in 24hr}}{\text{output KWh in 24hr} + [P_{cu}(KWh) + P_i(KWh)]\text{in 24hr}}}$$

## 8. PARALLEL OPERATION OF TRANSFORMERS

**Conditions for parallel operation of Single-Phase transformers:**

Necessary conditions:

- The transformers must have the same polarities.
- The transformers should have equal turn ratios.

Desirable conditions:

- The voltages at full load across transformers internal impedance should be equal.
- The ratios of their winding resistances to reactance should be equal for both transformers. This condition ensures that both transformers operate at the same power factor, thus sharing active power and reactive volt amperes according to their ratings.

**Conditions for parallel operation of Three-Phase transformers:**

The condition for the parallel operation of single phase and three phase transformers is the same but with the following additions:

- The phase sequence of the transformers must be identical.
- The primary and secondary voltages of all the transformers connected in parallel must have the same phase shift.

i.e., Leakage impedance $\propto \frac{1}{\text{kVA rating}}$

Let two transformers A and B having $Z_{ea}$ and $Z_{eb}$ equivalent impedances are in parallel and $S_{ar}$ and $S_{br}$ are the kVA rating respectively.

$$Z_{ea} \propto \frac{1}{S_{ar}}$$

$$Z_{eb} \propto \frac{1}{S_{br}}$$

$$\Rightarrow \quad \frac{Z_{ea}}{Z_{eb}} = \left(\frac{S_{br}}{S_{ar}}\right)$$

$$\therefore \quad \frac{I_a}{I_b} = \left(\frac{S_{ar}}{S_{br}}\right)$$

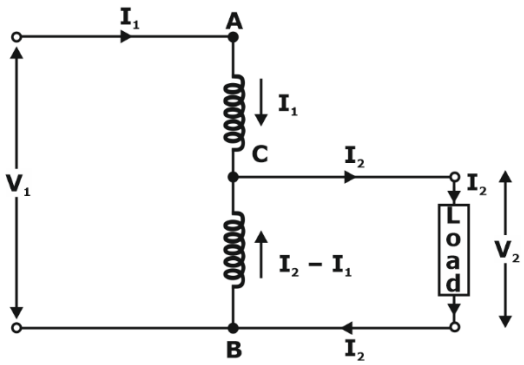
Hence, kVA shared by $A = \left(\frac{Z_{eb}}{Z_{ea} + Z_{eb}}\right) S$

and kVA shared by $B = \left(\frac{Z_{ea}}{Z_{eb} + Z_{ea}}\right) S$

where, S = Total kVA

**AUTO – TRANSFORMER**

An autotransformer is a transformer whose winding is common to both primary and secondary circuits. The primary and secondary are connected electrically as well as coupled magnetically.




**Auto-transformer construction**

- K of auto-transformer = $\frac{LV}{HV}$

  $(kVA)_{induction} = (V_1 - V_2)I_1$

  Input kVA = $V_1I_1$

  $$\frac{(kVA)_{induction}}{\text{input KVA}} = \frac{(V_1 - V_2)I_1}{V_1I_1} = 1 - \frac{LV}{HV} = 1 - K$$

  $\therefore$ $(kVA)_{induction}$ = (1 – K) input kVA

  $(kVA)_{conduction}$ = Input kVA – $(kVA)_{ind}$

  $(kVA)_{conduction}$ = K × input kVA
- Weight of conductor in section AB of auto t / f $\propto (N_1 - N_2)I_1$
- Weight of conductor in section BC of auto t / f $\propto (I_1 - I_2)N_1$

  $\therefore$ Total Weight of conductor in auto t/f is

  $\propto I_1(N_1 - N_2)N_1 + (N_2 - N_1)N_2$

  $\propto 2(N_1 - N_2)I_1$
- Total Weight of conductor in two winding transformer

  $\propto I_1N_1 + I_2N_2 \propto 2I_1N_1$
- $$\frac{\text{Wt. of conductor in an auto t / f}}{\text{Wt. of conductor in 2 wdg t / f}} = \frac{2(N_1 - N_2)I_1}{2N_1I_1} = 1 - \frac{N_2}{N_1} = 1 - K$$

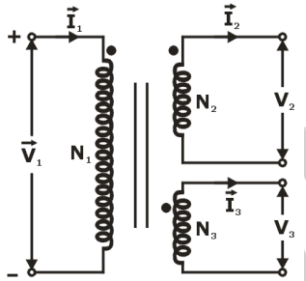
  Weight of conductor in auto-transformer = (1 – K) (Weight of conductor in two winding transformers)
- Thus, saving of conductor material if auto-transformer is used = K × (Conductor weight in two winding transformer)

**Some Important Relations:**

- $$\frac{\text{Weight of conductor in auto transformer}}{\text{Weight of conductor in 2-winding transformer}} = (1 - k)$$

- $\frac{\text{kVA rating as an auto transformer}}{\text{kVA rating as 2-winding transformer}} = \left(\frac{1}{1-k}\right)$
- $\frac{\text{p.u. full load losses as an auto transformer}}{\text{p.u. full load losses as a 2-winding transformer}} = (1-k)$
- $\frac{\text{p.u. impedance drop in an auto transformer}}{\text{p.u. impedance drop in 2-winding transformer}} = (1-k)$
- $\frac{\text{Voltage regulation as in auto transformer}}{\text{Voltage regulation in 2-winding transformer}} = (1-k)$

**TERTIARY WINDING**




**Tertiary winding of transformer**

$$\frac{\vec{V}_1}{\vec{N}_1} = \frac{\vec{V}_2}{\vec{N}_2} = \frac{\vec{V}_3}{\vec{N}_3}$$

MMF balance:

$$N_1\vec{I}_1 - N_2\vec{I}_2 - N_3\vec{I}_3 = N_1\vec{I}_0$$

$$\vec{I}_1^{\,*} = \vec{I}_0^{\,*} + \frac{N_2}{N_1} \times \vec{I}_2^{\,*} + \frac{N_3}{N_1} \times \vec{I}_3^{\,*}$$

$$\vec{V}_1\vec{I}_1 * = \vec{V}_1\vec{I}_0 * + \left(\vec{V}_1 \times \frac{N_2}{N_1}\right) \times \vec{I}_2 * + \left(\vec{V}_1 \times \frac{N_3}{N_1}\right) \times \vec{I}_3 *$$

$$\vec{V}_1 I_1 * = \vec{V}_1\vec{I}_0 * + \vec{V}_2\vec{I}_2 * + \vec{V}_3\vec{I}_3 *$$

$$\boxed{\vec{S}_1 = \vec{S}_2 + \vec{S}_0 + \vec{S}_3}$$

**Application of Different 3-phase Connections**:

1. Δ/Δ → This connection is used where we have simple 3-φ load, especially at low voltage levels. No mix loading is possible.

2. Y/Δ → This connection is used for step down application.

3. Δ/Y → This connection is used for step up application except in distribution system where, Δ/Y is used for step down application for mix loading.

4. Y/Y → This connection is quite attractive for HV applications. It is not generally used without a tertiary Δ.

**Impact of dimensions on varius parameters** :

Voltage rating of transformer depends on the level of insulation of the conductor.

Voltage rating is directly proportional to the area of the transformer.

Voltage rating is directly proportional to square of the dimensions of the transformer.

The transformer's current rating depends on the cross sectional area of the conductor.

Current rating is directly proportional to the area of the transformer.

Current rating is directly proportional to square of the dimensions of the transformer.

KVA rating of the transformer is quartic of the dimensions of the transformer.

## DC Machines

### EMF EQUATION AND DEVELOPED TORQUE

**Notations:**

ø → Flux/pole (Wb)

P → No. of poles

Z → Total no of conductors

N → Rotor speed (rpm)

A → Number of parallel paths

A = 2, for wave windings

A = P, for lap winding

$$E = \frac{P\phi NZ}{60A} \text{ Volts}$$

As we know that,

$$\omega_m = \frac{2\pi N}{60} \text{rad / sec}$$

$$N = \frac{60\omega}{2\pi}$$

Put this value in the above formula-

$$E = \frac{P\phi Z}{60A} \times \left(\frac{60\omega_m}{2\pi}\right)$$

$$E = \left(\frac{PZ}{2\pi A}\right)\phi\,\omega_m$$

$$\boxed{E_a = k\phi\omega_m} \text{ Where } k = \frac{PZ}{2\pi A} = \text{Machine constant}$$

**Developed Torque**

Developed power $P_a = E_a I_a$ and $P_a = T.\omega_m$

$\therefore P_a = T.\omega_m$

$$T = \frac{E_a I_a}{\omega_m} = k\phi I_a \quad \therefore E = k\varphi\omega_m$$

$$\boxed{T = k\phi I_a}$$ Induced emf ⇒ $\boxed{E_a = k\phi\omega_m}$

Developed Torque ⇒ $\boxed{T_d = k\phi I_a}$

**COMMUTATION PROCESS**

The reversal of current in the armature coil by means of brush and commutator bars, is called commutation process. Good commutation means no sparking at the brushes and with commutator surface remaining unaffected during continuous operation of the DC machine.

- Under commutation: $T_{commutation} > T_C$
- Over commutation: $T_{commutation} < T_C$

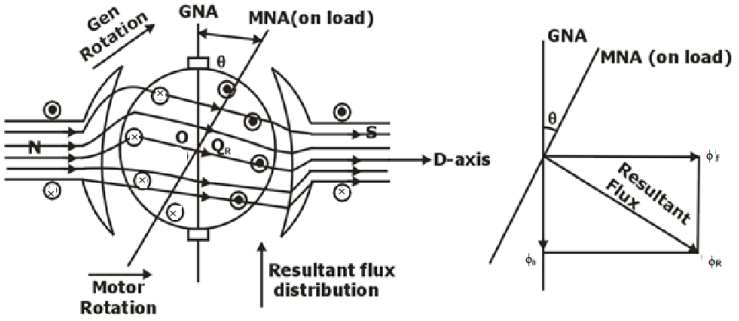
The time required by the coil current to change from $+I_C$ to $-I_C$ is called the commutation period $T_C$.

$$T_C = \frac{\text{Brush width}}{\text{Commutator peripheral speed}}$$

Where, $T_C$ is commutation period.

## ARMATURE REACTION

The effect of armature flux due to armature current over main field flux is known as Armature reaction effect.




**Armature Reaction Effect**

### Effects of Armature Reaction

(i) If strengthening effect = weakening effect, the average flux under the pole remains the same, and therefore, there is no demagnetizing effect of AR.
But due to saturation, strengthening effect is less than the weakening effect so average flux under each pole reduces and hence Armature reaction is "Demagnetizing".
(ii) The MNA shifts in the direction of rotation for generator action and in a direction opposite to rotation for motor.
(iii) Armature flux is perpendicular to field flux. So, the nature of armature reaction is cross magnetizing.

## INTERPOLES

- In large machines, interpoles, called commutating poles, are used to overcome commutation problems.
- The interpolar winding is designed to neutralize armature MMF in interpolar region.
- The interpoles winding carries the armature current as it is connected in series with the armature winding. The presence of interpoles ensures sparkless linear commutation.

## COMPENSATING WINDING

compensating winding consists of conductors embedded in pole faces and carry armature current in a direction opposite to armature conductor current under one pole arc. The

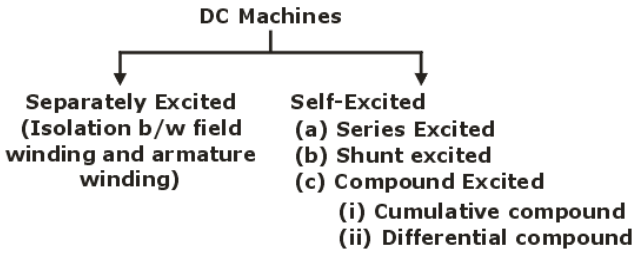
compensating winding may be designed to completely neutralize the armature MMF of the conductors that lie under the pole, resulting in restoration of main field flux.

DC machine compensating winding mmf per pole is:

- $F_{comp}(MMF) = \left[\frac{Z/2}{P} \times \left(\frac{\text{Pole arc}}{\text{Pole Pitch}}\right) \times \frac{I_a}{A}\right] AT/pole$
- $F_{comp}(MMF)_{Interpole} = \left[\frac{Z/2}{P} \times \left(1 - \frac{\text{Pole arc}}{\text{Pole Pitch}}\right) \times \frac{I_a}{A} + \frac{B_{interpole}}{\mu_0} \times L_{interpolar}\right] AT/pole$

**CLASSIFICATION OF DC MACHINES:**

DC Machines have been classified into multiple categories based on the connection of field winding and armature winding.




**Shunt Generator:**

- For a shunt generator with armature induced voltage $E_a$, armature current $I_a$ and armature resistance $R_a$, the terminal voltage V is:

$V = E_a - I_aR_a$

- The field current $I_f$ for a field resistance $R_f$ is: $I_f = \frac{V}{R_f}$

- The armature induced voltage $E_a$ and torque T with magnetic flux $\Phi$ at angular speed $\omega$ are:

$E_a = k_f\Phi\omega_m = k_m\omega_m \qquad T = k_f\Phi I_a = k_m I_a$

Where, $k_f$ and $k_m$ are design coefficients of the machine.

Note that for a shunt generator:

- Induced voltage is proportional to speed.
- Torque is proportional to armature current.

- The airgap power $P_e$ for shunt generator is:

$P_e = \omega_m T = E_a I_a = k_m\omega_m I_a$

**Series Generator:**

- For a series generator with armature induced voltage $E_a$, armature current $I_a$, armature resistance $R_a$ and field resistance $R_f$, the terminal voltage V is:

$V = E_a - (I_aR_a + I_aR_f) = E_a - I_a(R_a + R_f)$

The field current is equal to the armature current.

- The armature induced voltage $E_a$ and torque T with magnetic flux $\Phi$ at angular speed $\omega_m$ are:

$$E_a = k_f \Phi \omega_m I_a = k_m \omega_m I_a$$

$$T = k_f \Phi I_a^2 = k_m I_a^2$$

Where $k_f$ and $k_m$ are design coefficient of the machine.

Note that for a series generator:

- Induced voltage is proportional to both speed and armature current,
- Torque is proportional to the square of armature current.
- Armature current is inversely proportional to speed for a constant $E_a$

- The airgaps power $P_e$ for a series generator is:

$$P_e = \omega_m T = E_a I_a = k_m \omega_m I_a^2$$

**Cumulatively compounded DC generator: (long shunt):**

(a) $I_a = I_f + I_L$

(b) $V_t = E_a - I_a (R_a + R_s)$

(c) $I_{sf} = \frac{V_x}{R_f} =$ Shunt field current

(d) The equivalent effective shunt field current for this machine is given by

$$I_{sf} = I_{sf} + \frac{N_{se}}{N_f} I_a - \left( \frac{\text{Armature reaction MMF}}{N_f} \right)$$

Where, $N_{se}$ = No. of series field turns

$N_f$ = No of shunt field turns

**Differentially compounded DC generator: (long shunt):**

(a) $I_a = I_f + I_L$

(b) $V_t = E_a - I_a (R_a + R_s)$

(c) $I_{sf} = \frac{V_x}{R_f} =$ Shunt field current

(d) The equivalent effective shunt field current for this machine is given by

$$I_{sf} = I_{sf} - \frac{N_{se}}{N_f} I_a - \left( \frac{\text{Armature reaction MMF}}{N_f} \right)$$

Where, $N_{se}$ = No. of series field turns

$N_f$ = No of shunt field turns

**Shunt Motor:**

- For a shunt generator with armature induced voltage $E_a$, armature current $I_a$ and armature resistance $R_a$, the terminal voltage V is:

  $V = E_a + I_aR_a$

  The field current $I_f$ for a field resistance $R_f$ is: $I_f = \frac{V}{R_f}$

- The armature induced voltage $E_a$ and torque T with magnetic flux $\Phi$ at angular speed $\omega$ are:

  $E_a = k_f\Phi\omega = k_m\omega$

  $T = k_f\Phi I_a = k_m I_a$

  Where, $k_f$ and $k_m$ are design coefficients of the machine.

  Note that for a shunt motor:

  - Induced voltage is proportional to speed.
  - Torque is proportional to armature current.

- The airgap power $P_e$ for shunt generator is:

  $P_e = \omega T = E_aI_a = k_m\omega I_a$

- The speed of the shunt motor, $\omega = \frac{V}{k\phi} - \frac{TR_a}{(K\phi)^2}$ where, $K = \frac{PZ}{2\pi A}$

**Series Motor:**

- For a series motor with armature induced voltage $E_a$, armature current $I_a$, armature resistance $R_a$ and field resistance $R_f$, the terminal voltage V is:

  $V = E_a + (I_aR_a + I_aR_f) = E_a + I_a(R_a + R_f)$

  The field current is equal to the armature current.

- The armature induced voltage $E_a$ and torque T with magnetic flux $\Phi$ at angular speed $\omega$ are:

  $E_a = k_f\Phi\omega I_a = k_m\omega I_a \quad T = k_f\Phi I_a^2 = k_m I_a^2$

  Where $k_f$ and $k_m$ are design coefficient of the machine.

  Note that for a series motor:

  - Induced voltage is proportional to both speed and armature current.
  - Torque is proportional to the square of armature current.
  - Armature current is inversely proportional to speed for a constant $E_a$

- The airgaps power $P_e$ for a series motor is:

  $P_e = \omega T = E_aI_a = k_m\omega I_a^2$

**Losses:**

- Constant losses ($P_k$) = $P_{w\text{-}f}$ + $P_{io}$

  where, $P_{io}$ = No-load core loss

  $P_{w\text{-}f}$ = Windage and frictional losses

- Variable losses ($P_v$) = $P_c$ + $P_{st}$ + $P_b$

Where, $P_c$ = Copper losses = $I_a^2R_a$

$P_{st}$ = Stray load loss

$P_b$ = Brush contact drop = $V_b I_a$, where, $V_b$ = Brush voltage drop

- The total machine losses, $P_L = P_k + P_v$

**Efficiency:**

- The per unit efficiency η of an electrical machine with input power $P_{in}$, output power $P_{out}$ and power loss $P_{loss}$ is:

$$\eta = \frac{P_{out}}{P_{in}} = \frac{P_{out}}{(P_{out} + P_{loss})} = \frac{(P_{in} - P_{loss})}{P_{in}}$$

- Rearranging the efficiency equations:

$$P_{in} = P_{out} + P_{loss} = \frac{P_{out}}{\eta} = \frac{P_{loss}}{(1-\eta)}$$

$$P_{out} = P_{in} - P_{loss} = \eta P_{in} = \frac{\eta P_{loss}}{(1-\eta)}$$

$$P_{loss} = P_{in} - P_{out} = (1-\eta)P_{in} = \frac{(1-\eta)P_{out}}{\eta}$$

**Note :**

- The terminal voltage of the DC generator is given by $V_t = E_a - I_a R_a$
- The terminal voltage of the DC motor is given by $V_t = E_a + I_a R_a$
- Speed regulation of DC machine is given by, $SR = \frac{\omega_{nl} - \omega_{fl}}{\omega_{fl}} \times 100\% = \frac{N_{nl} - N_{fl}}{N_{fl}} \times 100\%$
- Voltage regulation, $VR = \frac{V_{nl} - V_{fl}}{V_{fl}} \times 100\%$

**OPERATING CHARACTERISTICS OF DC GENERATOR**

1. No-load characteristics:

$E_a = f(I_f)$, with constant speed.

This is also called **open-circuit characteristics or magnetization curve**.

2. Load Characteristics:

$V_t = f(I_f)$, with both $I_a$ & N constant.

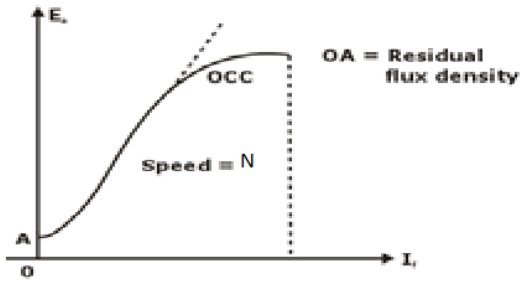
This is called as load magnetization curve.

3. External characteristics:

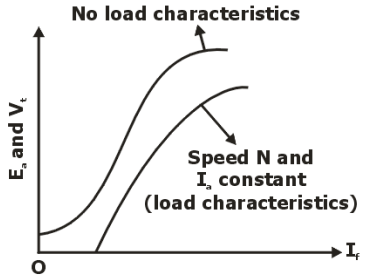
$V_t = f(I_L)$, with both $I_f$ & N constant.

A. **Separately excited generators**

1. No-load characteristics:

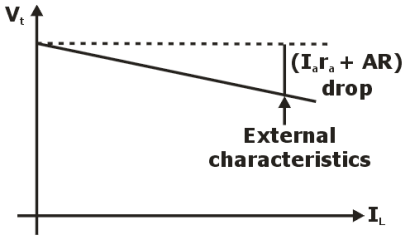



2. Load characteristics



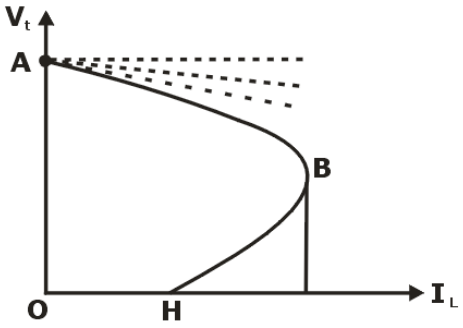

3. External characteristics

$V_t = E_a - I_aR_a$




**B. Shunt Generator:**

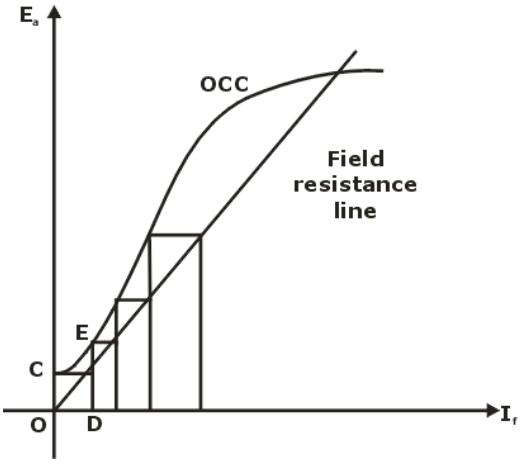
External characteristics




Terminal voltage falls due to AR drop & $I_ar_a$ drop due to which $I_f$ decreases which leads to further decrease in $V_t$.

**Voltage build up in Shunt generator:**

The voltage build up in shunt generator requires some residual flux and emf for voltage build up.



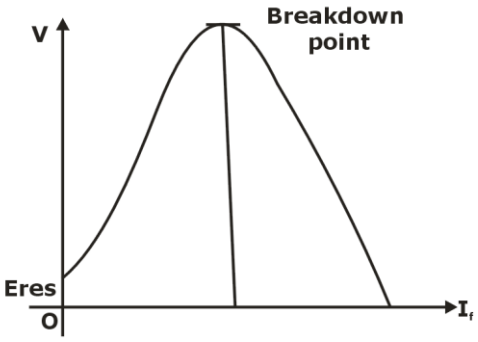

- Due to residual flux, emf OC is induced in armature.
- Due to this induced emf field current OD flows in the circuit.
- This causes an emf DE to be induced as per OCC.
- This further causes a current to flow in field winding.
- This effect goes on increasing till the field resistance line cuts OCC.

**Necessary Conditions for Voltage Build up:**

1. There must be a residual flux.
2. Correct the polarity of field winding with armature winding so that field flux aids residual flux.
3. Field resistance must be less than critical value.
4. Speed of rotation should be more than critical value for given $R_f$.

**C. Series generator:**

External characteristics:




As load resistance is reduced, armature current or field current increases which leads to increase in flux & hence induced voltage also increases upto breakdown point. After that ∅ becomes constant, at saturation, So $E_a$ becomes constant and due to increase in armature current terminal voltage reduces.

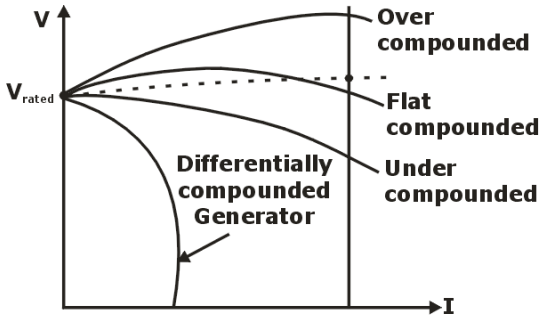
**D. Compound Generator:**

Cumulative compound generators are more common than differentially compound generators as they have almost constant voltage from no load to full load.

In cumulative compound generator series flux aids shunt flux and depending upon number of series field turns,

- Under-compounded
- Flat- compounded
- Over-compounded

In differentially compounded generator, Series flux opposes shunt flux and thus $V_t$ falls more rapidly with load.




**Different generator characteristics keep the no load voltage same in all cases.**

**OPERATING CHARACTERISTICS OF DC MOTOR:**

The following characteristics are-

- Speed-Armature current characteristics
- Torque-Armature current characteristics
- Speed-Torque characteristics

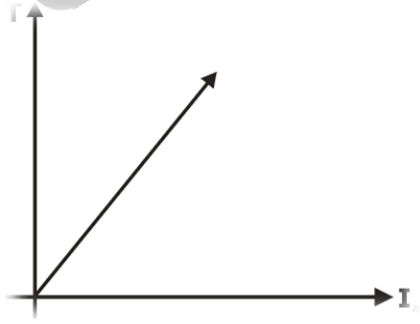
**A. Separately excited DC motor**

1. Torque-Armature current characteristics

Developed torque in a DC machine depends on Armature current as shown below-

$T = K\emptyset I_a$

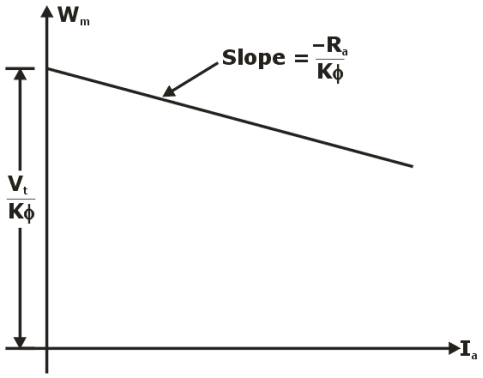
$\boxed{T \propto I_a}$ $\emptyset$ = constant




2. Speed-Armature current characteristics

$E_b = V_t - I_a R_a$

$$\omega_m = \frac{E_b}{K\phi} = \frac{(V_t - I_a R_a)}{k\phi}$$



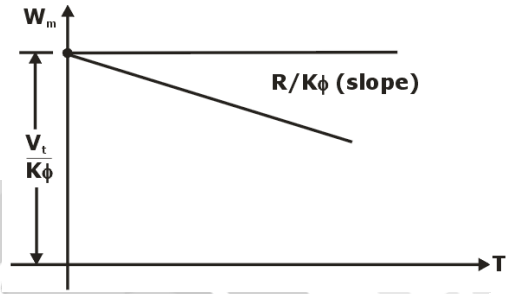

3. Speed-Torque characteristics:

As we know that,

$$\omega_m = \frac{V_t - I_a R_a}{k\phi} \quad \ldots (1)$$

$$T \propto I_a \quad \ldots (2)$$

$$\therefore \omega_m \propto \frac{V_t}{k\phi} - T\frac{R_a}{k\phi}$$




**Note:**

- For shunt excited DC motor all characteristics are same as that of separately excited DC motor.

**B. Series excited DC motor:**

1. Speed-Armature current characteristics

For series motor, flux is proportional to armature current.

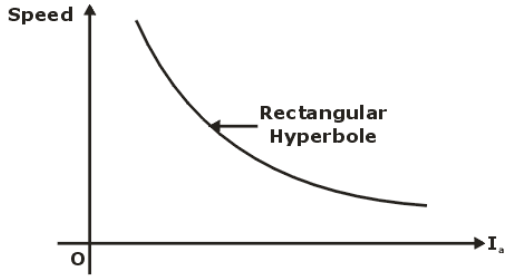
$$\phi \propto I_a$$

$$E_b = V_t - I_a(r_a + r_{se})$$

$$\omega_m = \frac{V_t}{k\phi} - \frac{I_a(r_a + r_{se})}{k\phi}$$

$$\therefore \quad \boxed{E_b = k\phi\omega_m}$$

$$\boxed{\omega_m = \frac{V_t}{kI_a} - \frac{(r_a + r_{se})}{k}}$$




2. Speed-Torque characteristic

$T = k_\varphi I_a$

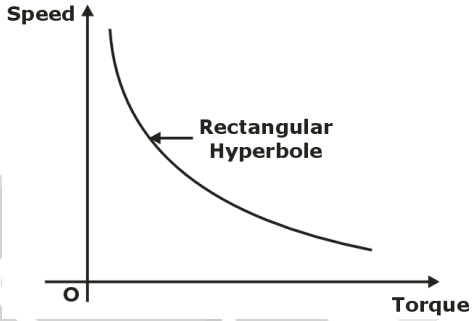
$$\boxed{T = kI_a^2}$$

$\therefore \phi \propto I_a$

$$I_a = \sqrt{\frac{T}{K}} \Rightarrow I_a \propto \sqrt{T}$$

$$\therefore \omega_m = \frac{V_t}{kI_a} - \frac{(r_a + r_{se})}{k}$$

$$\boxed{\omega_m = \frac{V_t}{k\sqrt{T}} - \frac{(r_a + r_{se})}{k}}$$



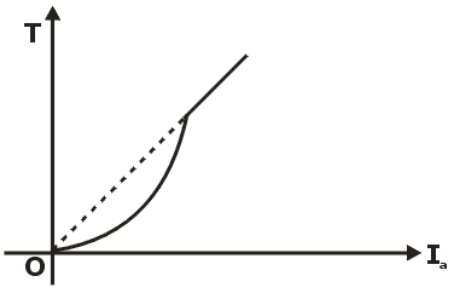

3. Torque-Armature Current characteristics

$\therefore$ flux is proportional to armature current in series excited DC motor.

$\therefore$ $T = k\varphi I_a$

$$\boxed{T = kI_a^2}$$

For large $I_a$ as $\phi \simeq$ constant so $T \propto I_a$




**C. DC compound Motor**

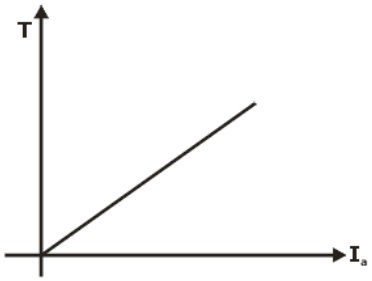
- Cumulatively compound motor

The series field aids the armature field in the motor,

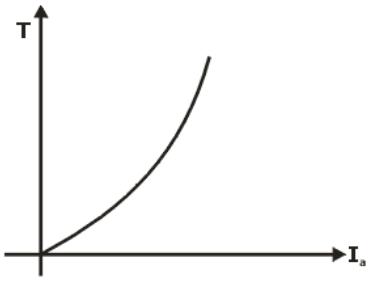
$Ø_r = Ø_{se} + Ø_{sh}$

If series field dominates, characteristics will look like series motor; otherwise, shunt motor in the case of dominant shunt field.

1. Torque-Armature current characteristics

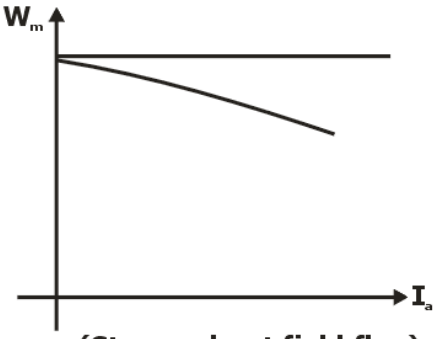



(Strong shunt field flux)

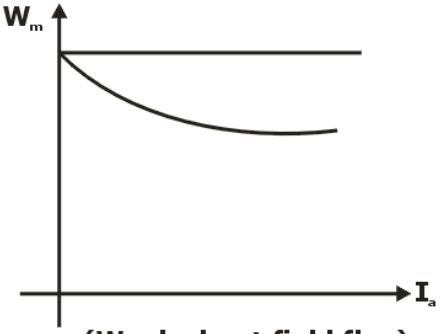



(Weak shunt field flux)

2. Speed-Armature current characteristics

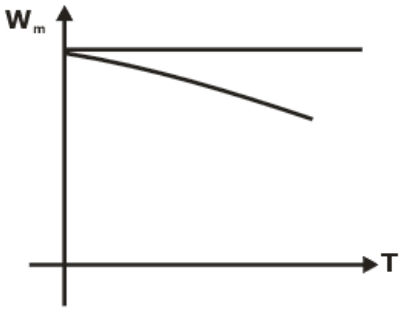



(Strong shunt field flux)

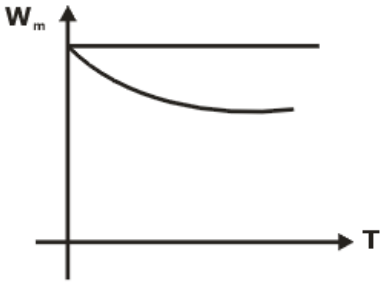



(Weak shunt field flux)

3. Speed-torque characteristics




(Strong shunt field flux)



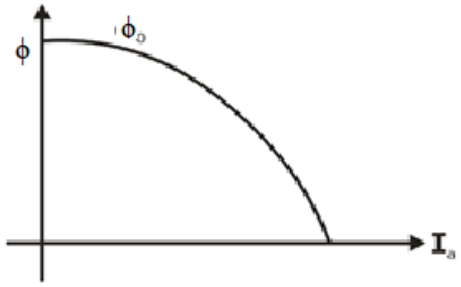

(Weak shunt field flux)

- Differentially compound motor

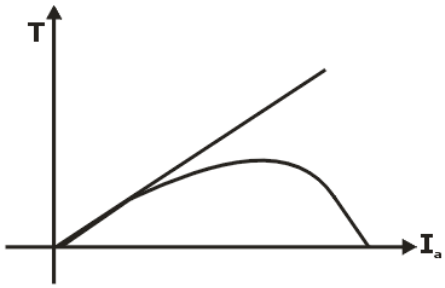
The series flux will oppose the shunt field flux.ø

$$\boxed{\phi_r = \phi_{sh} - \phi_{se}}$$

The flux decreased as the armature current or field current is increased.

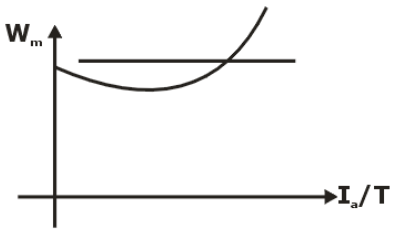



1. Torque-Armature current characteristics




$$T \propto \Phi_r$$

2. Speed-armature current/Speed-Torque characteristics




$$\omega \propto \frac{1}{\phi}$$

$$\phi \propto I_a \propto T$$

The only DC motor which can have zero speed regulation.

**SPEED CONTROL OF DC MOTOR:**

For variable speed DC drive the speed control of DC motor is necessary. As the speed of DC motor is given by:

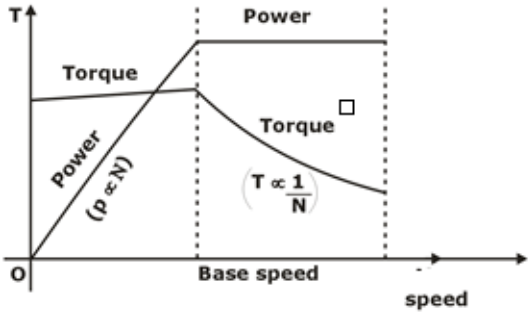
$$\omega_m = \frac{V_t - I_a r_a}{k\phi} \quad \ldots(1)$$

Based on the above expression, there are 3 methods of speed control.

1. Variation of resistance in armature circuit.
2. Variation of field flux
3. Variation of armature terminal voltage

**Ward Leonard system**

- With armature voltage controlled method constant torque and variable power drive is obtained from speed below the base speed. (constant torque drive)
- With the field control method, constant power and variable torque is obtained for speed above base speed. (constant power drive)




**Constant power and Torque Drive**

- Below base speed machine acts as constant torque drive. ( $P \propto N$ )
- Above base speed machine acts as constant power drive. $\left(T \propto \frac{1}{N}\right)$

| S.No. | | Lap winding | Wave winding |
|---|---|---|---|
| 1. | Coil span: | $Y_{cs} = s/p$ | $Y_{cs} = s/p$ |
| 2. | Back pitch | $Y_b = UY_{cs}$ | $Y_b = UY_{cs}$ |
| 3. | Commutator pitch | $Y_c = 1$ for progressive winding<br>$Yc = -1$ for retrogressive winding | $Y_c = \frac{2(c+1)}{p}$ for progressive winding<br>$Y_c = \frac{2(c+1)}{p}$ for retrogressive winding<br>($Y_c$ must be integer) |
| 4. | Front pitch | $Y_f = Y_b + 2$ for progressive winding<br>$Y_f = Y_b - 2$ for retrogressive winding | $Y_f = 2Y_b - Y_b$ |
| 5. | Parallel paths | A = P | A = 2 |
| 6. | Conductor current | $I_c = \frac{I_a}{A}$ | $I_c = \frac{I_a}{A}$ |
| 7. | No. of brushes | No. of brushes = A = P | No. of brushes = 2 |

Where,

S = No. of commutator segments

P = No. of poles

U = No. of coil sides/No. of poles = $\frac{2c}{p}$

C = No. of coils on the rotor

A = No. of armature parallel paths

$I_a$ = Armature current

## Induction Machines

**Types of IM:**

There are two types of rotor construction in induction machine:

(i) Squirrel Cage rotor IM

(ii) Slip ring rotor IM

**IMPORTANT DEFINITIONS**

1. **Slip:**

   It is the speed difference between the synchronous speed and the rotor speed expressed as the fraction of synchronous speed $N_s$.

   $$s = \left(\frac{N_s - N}{N_s}\right)$$

   **Note:** A well-designed induction motor runs at very low slip value at no-load. Its operating range is also a low value of around 1% to 5% from no load to full load.

2. **Slip Speed:**

   It is the speed difference between the stator (synchronous speed) and the rotor speed.

   Slip speed, $sN_s = N_s - N$

3. **Frequency of Rotor Current or E.M.F:**

   Rotor frequency, $f_r = sf$

Where, s = slip

f = supply frequency

**4. Rotor E.M.F:**

The rotor emf is directly proportional to the frequency. At standstill, this frequency is equal to supply f but at running condition it is equal to (sf).

Hence, induced emf in the rotor at running condition will be s-times the induced emf at standstill.

$E'_2 = \sqrt{2}\pi(sf)k_w f = s\,E_2$

**5. Rotor inductance ($X_2$):**

At starting, $f_r = f$

$X_2 = 2\pi f L_2$

At running, $f_r = sf$

$X'_2 = 2\pi sf L_2 = sX_2$

$X'_2 = sX_2$

**6. Rotor power factor:**

At starting,

$$\text{Cos}\phi = \frac{R_2}{\sqrt{R_2^2 + X_2^2}}$$

At running,

$$\text{Cos}\phi' = \frac{R_2}{\sqrt{R_2^2 + (sX_2)^2}}$$

pf is better in running condition.

**7. Current in Rotor:**

At starting,

$$I_2 = \frac{E_2}{\sqrt{R_2^2 + X_2^2}}$$

At running.

$$I_2' = \frac{sE_2}{\sqrt{R_2^2 + (sX_2)^2}}$$

**COMPARISON BETWEEN INDUCTION MACHINE AND TRANSFORMER**

| Transformer | Induction Machine |
|---|---|
| 1. Electrical-to-electrical energy conversion through magnetic medium is high permeability core. | 1. Electrical to mechanical energy conversion through magnetic medium in the high reluctance air gap. |

| | |
|---|---|
| 2. Low magnetizing current required therefore, $I_0$ equals 2 – 6% of rated current. | 2. High magnetizing current is required therefore, $I_0$ equals around 30% of rated current. |
| 3. Concentrated winding does not produce RMF. | 3. Distributed winding produce RMF. |
| 4. Induce emf depends on turn ratio. | 4. Induce emf depends on turns ratio only at standstill. |
| 5. Frequency of primary & secondary is same. | 5. frequency of stator & rotor same only at stand still. |
| 6. Constant flux device under steady state. | 6. Constant flux Machine |
| 7. Mutual induction present | Mutual induction present |
| 8. Secondary is not short circuit but connected to electrical load. | 8. Rotor should be essentially short circuit and the load is mechanical across the shaft. |
| 9. No mechanical loss | 9. Contains mechanical losses |

## TORQUE IN INDUCTION MOTOR: (NEGLECTING STATOR IMPEDANCE)

The relative speed between stator field and rotor field is zero. The interaction of both fields generates a steady torque.

$$T_e \propto F_1 F_2 \cos \phi$$

where, $F_1 \rightarrow$ Stator field produced by induced voltage in stator ($E_2$)

$F_2 \rightarrow$ rotor field produced by induced current in rotor ($I_2'$)

$\phi \rightarrow$ angle between stator and rotor field

**Running torque:**

Let the motor run at speed N with the slip (s) & delivering full load torque '$T_f$'

$T_f \propto E_2 I_2' \cos \phi_2$

$$T_f = kE_2 \times \frac{sE_2}{\sqrt{R_2^2 + (sX_2)^2}} \times \frac{R_2}{\sqrt{R_2^2 + (sX_2)^2}}$$

$$T_f = K \frac{sE_2^2 R_2}{R_2^2 + (sX_2)^2}$$

For given voltage,

$$T_f \propto \frac{sR_2}{R_2^2 + (sX_2)^2}$$

**Condition for maximum running torque:**

$$\frac{dT_s}{ds} = 0$$

$R_2 = sX_2$

Slip at which maximum torque ($T_m$) occurs $s_m$:

$$s_m = \frac{R_2}{X_2}$$

By putting the value of $s_m$ in running torque.

$$T_{max} \propto \frac{1}{2X_2}$$

**Note:** $T_{max}$ is independent to rotor resistance ($R_2$), varying $R_2$ will shift the point of maximum torque.

**Starting Torque:**

At starting, slip is 1

By putting slip value equal to 1 in running torque

$$T_{st} = K\frac{E_2^2 R_2}{R_2^2 + X_2^2}$$

**Condition for maximum starting torque:**

$$\frac{dT_{st}}{dR_2} = 0$$

$$\Rightarrow R_2 = X_2$$

# If the rotor resistance is made equal to its reactance at stand still than the motor starts with maximum starting torque.

# Under such conditions the rotor power factor will be 0.707 lag.

**Motor Torque in terms of $T_{em}$:**

- The torque expression of an induction motor can also be expressed in terms of maximum torque $T_{em}$ and dimension less ratio $\frac{S}{S_{mT}}$. To get a simple and approximate expression, stator resistance $r_1$ or the stator equivalent resistance $R_e$ is neglected.

$$\therefore \frac{T_e}{T_{em}} = \frac{2X}{\left(\frac{r_2}{s}\right)^2 + X^2} \times \frac{r_2}{s}$$

- The slip at which maximum torque occurs is

$$S_{mT} = \frac{r_2}{X} \qquad \therefore r_2 = S_{mT}X$$

$$\therefore \frac{T_e}{T_{em}} = \frac{2X}{\left[\frac{S_{mT}X}{s}\right]^2 + X^2} \times \frac{S_{mT}X}{s}$$

$$\Rightarrow \frac{T_e}{T_{em}} = \frac{2}{\frac{S_{mT}}{s} + \frac{S}{s_{mT}}}$$

$$T_e = \frac{2T_{em}}{\frac{S_{mT}}{s} + \frac{S}{s_{mT}}}$$

## TORQUE SLIP CHARACTERISTIC

Torque equation is given by:

$$T_f = K\frac{sE_2^2R_2}{R_2^2 + (sX_2)^2}$$

**Low slip region:**

$$T_f \propto \frac{sR_2}{R_2^2 + (sX_2)^2}$$

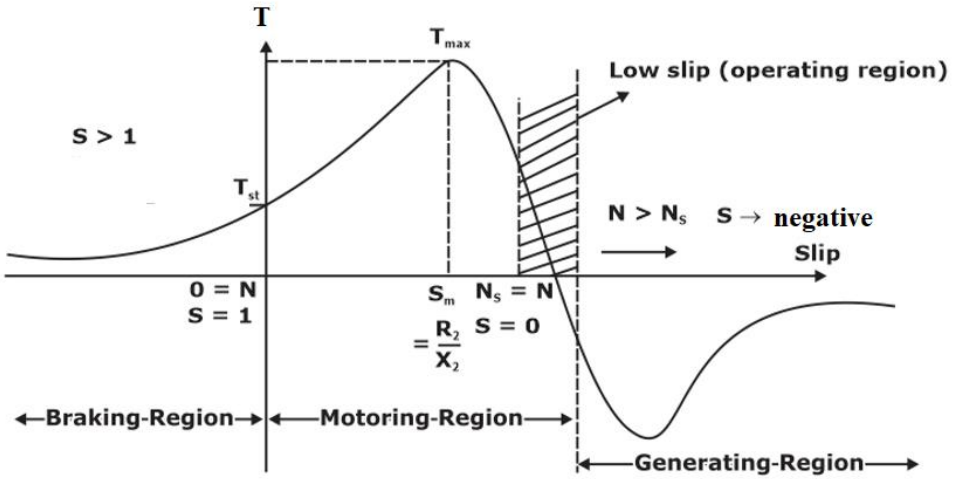
For low slip, $(sX_2)^2 <<< R_2^2$

$$T_f \propto \frac{s}{R_2} \Rightarrow T_f \propto s$$

**For high slip region:**

$(sX_2)^2 >>> R_2^2$

$$T_f \propto \frac{R^2}{sX_2^2} \Rightarrow T_f \propto \frac{1}{s}$$




**Torque-slip characteristic on induction machine**

In an operating region, the torque-slip characteristic is essentially a straight line. It has good speed regulation.

**Speed Regulation:** It is the change in speed when the full load across shaft is disconnected. It is expressed in % of full load speed.

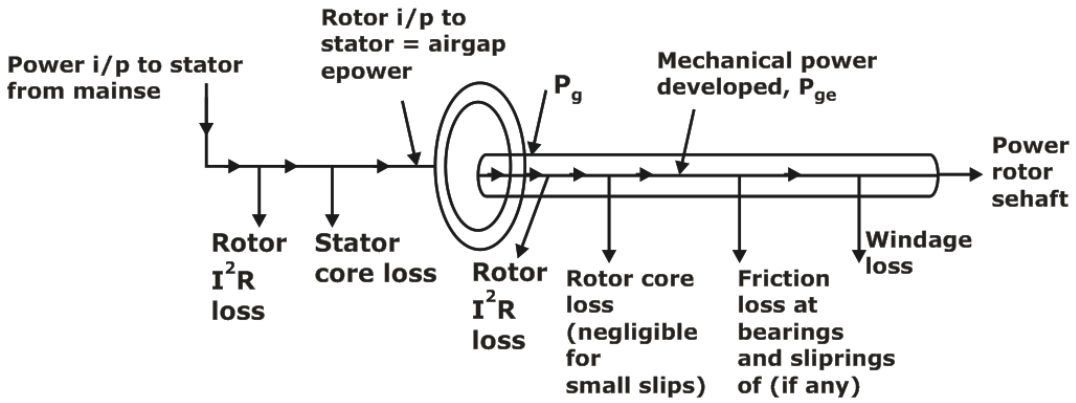
$$S.R\% = \frac{N_o - N}{N} \times 100$$

**Note:** As the rotor resistance increases under operating region, to maintain constant torque, slip also increases correspondingly.

Constant torque, $T_f \propto \frac{\uparrow s}{\uparrow R_2}$

## POWER FLOW IN INDUCTION MOTOR

The Power Flow Diagram of an Induction Motor is shown below.




**Power Flow Diagram of Induction Motor**

- The power flow diagram of $3-\phi$ induction motor is:
- Per phase power input to rotor is

  $P_g = E_2 I_2 \cos\theta_2$

  $$\cos\theta_2 = \frac{\text{Per phase rotor resistance}}{\text{Per phase rotor impedance}} = \frac{r_2/s}{\sqrt{(r_2/s)^2 + (x_2)^2}}$$

  $$\therefore P_g = E_2 I_2 \times \frac{r_2/s}{\sqrt{(r_2/s)^2 + (x_2)^2}} = \frac{E_2}{\sqrt{(r_2/s)^2 + (x_2)^2}} \times I_2 \frac{r_2}{s} = I_2^2 \frac{r_2}{s}$$

- $P_g$ is the power transferred from stator to rotor across the air gap. Therefore, $P_g$ is called air gap power

  $$P_g = I_2^2 \frac{r_2}{s} = I_2^2 r_2 + I_2^2 r_2 \left[\frac{1-s}{s}\right]$$

  $P_g$ = (Rotor ohmic loss) + Internal mechanical power developed in rotor ($P_m$) = $sP_g$ +(1 – s)$P_g$

  $$\therefore P_m = (1-s)P_g = I_2^2 r_2 \left[\frac{1-s}{s}\right]$$

Rotor ohmic loss = $\left[\frac{s}{1-s}\right]P_m = sP_g$

- Internal (or gross) torque developed per phase is given by

$$T_e = \frac{\text{Internal mechanical power developed in rotor}}{\text{Rotor speed in mechanical radian per sec}}$$

$$T_e = \frac{P_m}{\omega_r} = \frac{(1-S)P_g}{(1-S)\omega_s} = \frac{P_g}{\omega_s}$$

- Electromagnetic torque $T_e$ can also be expressed as

$$T_e = \frac{P_g}{\omega_s} = \frac{1}{\omega_s} \times \frac{I_2^2 r_2}{s} = \frac{\text{Rotor ohmic loss}}{(\omega_s)\,\text{slip}}$$

$$\therefore T_e = \frac{\text{Rotor ohmic loss}}{(\omega_s)\,\text{slip}}$$

- Power available at the shaft can be obtained from $P_g$ as follows. Output or shaft power, $P_{sh} = P_m$ – Mechanical losses
- Mechanical losses imply friction and windage losses.

  $P_{sh} = P_g$ – Rotor ohmic loss – Friction and windage losses = Net mechanical power output or net power output or shaft power

$$T_{sh} = \frac{P_{sh}}{\text{Rotor speed}} = \frac{P_{sh}}{(1-s)\omega_s}$$

- If the stator input is known. Then air gap power $P_g$ is given by $P_g$ = stator power input – stator $I^2R$ loss – stator core loss.
- Ratio of Rotor input power, rotor copper losses and gross mechanical output is

$$\frac{I_r^2 R_2}{s} : I_r^2 R_2 : I_r^2 R_2 \left[\frac{1}{s} - 1\right]$$

$$\Rightarrow 1 : s : (1-s)$$

$\therefore$ Rotor copper losses = s × Rotor input

Gross Mechanical output = (1–s) × Rotor input.

Rotor copper losses = (Gross mechanical output) × $\frac{s}{1-s}$

Efficiency of the rotor is approximately equal to

$$\eta_{rotor} = \frac{\text{Gross mechanical power output}}{\text{Rotor input}}$$

$$\frac{(1-s)\,\text{Rotor input}}{\text{Rotor input}} = 1 - s = 1 - \frac{N_S - N}{N_S} = \frac{N}{N_s}$$

$$\eta_{rotor} \simeq \frac{N}{N_s}$$

**DETERMINATION OF EQUIVALENT CIRCUIT PARAMETERS**

It requires two tests to be conducted on a given induction motor to find out its parameter.

(1) No-load test

(2) Block Rotor test

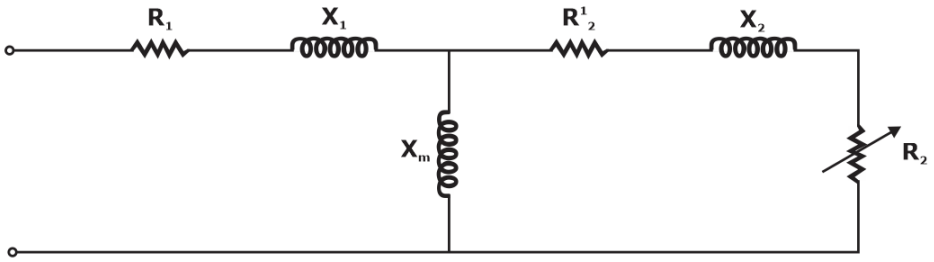
1. **No-Load Test**
   - This test is used to determine the core losses.
   - The motor is made to run at rated voltage and frequency on no load and stator voltage $V_{nl}$, input current $I_{nl}$ and input power $P_{nl}$ are recorded.
   - Rotor core loss and rotor copper loss are less and can be neglected because current in rotor at no load is much less.
   - $W_{NL} - 3I_0^2R_1$ = Constant losses
2. **Blocked Rotor Test:**

This test is done by blocking the rotor and applying small voltage of around (5 to 10%) across stator winding while ensuring rated current drawn by the motor.

**Note:** As voltage is low, core loss is neglected on stator and rotor.

$W_{BR}$ = stator copper loss + Rotor copper loss

$$W_{BR} = I_1^2R_{01} = I_{BR}^2R_{BR}$$




**Equivalent circuit at Blocked rotor test**

$$Z_{BR} = \frac{V_{BR}/\sqrt{3}}{I_{BR}}$$

$$X_{BR} = \sqrt{Z_{BR}^2 - R_{BR}^2}$$

On calculating:

$$X_1 = X'_2 = X_{BR}/2$$

$$R'_2 = (R_{BR} - R_1)\left(\frac{X_2 + X_m}{X_m}\right)^2$$

Hence, all the parameters of the equivalent circuit can be calculated by conducting these tests.

## STARTING OF THREE PHASE INDUCTION MOTORS

1. **Direct-on line (across the line) starting:**

- The relation between starting torque and full load torque is:

$$T_e = \frac{1}{\omega_s} \times I_2^2 \frac{r_2}{s}$$

$$\therefore \quad \frac{T_{e,st}}{T_{e,fl}} = \frac{I_{2,st}^2 \frac{r_2}{r_1}}{I_{2,fl}^2 \frac{r_2}{s_{r_1}}} = \left[\frac{I_{2,st}}{I_{2,fl}}\right]^2 \times s_{fl}$$

The above equation valid for rotor resistance, remains constant.

$$\frac{T_{e,st}}{T_{e,fl}} = \left[\frac{I_{st}}{I_{fl}}\right]^2 \times s_{fl}$$

Where,

$$\frac{I_{st}}{I_{fl}} = \frac{(\text{Effective rotor to stator turns ratio}) I_{2,st}}{(\text{Effective rotor to stator turns ratio}) I_{2,sl}}$$

- Per phase short–circuit current at stand still (or at starting) is,

$$I_{sc} = \frac{V_1}{z_{sc}}$$

Where, $Z_{sc} = (r_1 + r_2) + j(X_1 + X_2)$

Here shunt branch parameters of equivalent circuit are neglected.

- Therefore, for direct switching.

$$I_{st} = I_{sc} = \frac{V_1}{z_{sc}}$$

$$\therefore \frac{T_{est}}{T_{efl}} = \left[\frac{I_{sc}}{I_{fl}}\right]^2 s_{fl}$$

**2. Stator resistor (or reactor) starting:**

Since per phase voltage is reduced to xV, the per phase starting current $I_{st}$ is given by

$$I_{st} = \frac{xV_1}{z_{sc}} = xI_{sc}$$

As before $\therefore \frac{T_{est}}{T_{efl}} = \left[\frac{I_{sc}}{I_{fl}}\right]^2 s_{f1} = \left[\frac{xI_{sc}}{I_{fl}}\right]^2 s_{fl}$

- In an induction motor, torque $\propto (\text{Voltage})^2$

$$\therefore \frac{\text{Starting torque with reactor starting}}{\text{Starting torque with direct switching}} = \left[\frac{xV_1}{V_1}\right]^2 = x^2$$

**3. Auto transformer starting:**

- Per phase starting current from the supply mains is $I_{st} = x^2 I_{sc}$

$$\frac{T_{e,st}}{T_{e,fl}} = \frac{\text{Per phase starting current in motor winding}}{\text{Per phase motor full load current}} \times s_{fl}$$

$$\frac{T_{e,st}}{T_{e,fl}} = \frac{I_{st}I_{sc}}{I_{fl}^2} \times s_{fl}$$

$$\frac{T_{e,st}\text{with an auto transformer}}{T_{e,st}\text{with direct switching}} = \left[\frac{xV_1}{V_1}\right]^2 = x^2$$

**4. Star–Delta method of Starting:**

- $$\frac{\text{Starting torque with star delta starter}}{\text{Starting torque with direct switching in delta}} = \frac{\left[\frac{V_L}{\sqrt{3}}\right]^2}{[V_L]^2} = \frac{1}{3}$$

  $\therefore$ Star delta starter also reduces the starting torque to one-third of that produce by direct switching in delta.
- With star–delta starter, a motor behaves as if it were started by an auto transformer starter with $x = \frac{1}{\sqrt{3}} = 0.577$ i.e. with 57.7% tapping.
- $$\frac{\text{Starting torque with star delta starter, } T_{e,st}}{\text{Starting torque with direct switching in delta, } T_{efl,d}} = \frac{\frac{1}{\omega_s}\left(I_{st,y}\right)^2 \frac{r_2}{1}}{\frac{1}{\omega_s}\left(I_{st,d}\right)^2 \frac{r_2}{s_{fl}}}$$

$$\frac{\left[\frac{1}{\sqrt{3}} I_{st,d}\right]^2}{\left(I_{fl,d}\right)^2} \times S_{fl} = \frac{1}{3}\left[\frac{I_{sc}d}{I_{fl}d}\right]^2 \times s_{fl}$$

## SPEED CONTROL OF 3 – PHASE INDUCTION MOTORS

Running speed of an induction motor is given as:

$$N = \frac{120\ f}{P}(1 - s)$$

The method involved in speed control of induction motor is not simple and efficient.
It does not offer wide range of speed control.

**1. Line Voltage Control:**

As we known,

$T \propto sV^2$

- If voltage is reduced by keeping constant torque, the slip had to be increased. Hence, the speed decreases.
- It draws more current and does not offer wide range of speed control.

**2. V/f control:**

It is basically a frequency control.

To keep V/f ratio constant, it requires variable voltage also otherwise stator and rotor core gets saturated.

$$N_s = \frac{120f}{P} \Rightarrow N_s \propto f$$

$$s_m = \frac{R_2}{X_2} \Rightarrow s_m \propto \frac{1}{f} \text{ (as reactance depends on frequency)}$$

$$T_{max} = \frac{3 \times 60}{2\pi N_s} \frac{E^2}{2X_2}$$

$$T_{max} \propto \frac{V^2}{f \times f} \Rightarrow \text{constant}$$

(as $\frac{V}{f}$ is constant)

**3. Rotor resistance control:**

- Speed control by this method is used when speed is to be reduced for a short period.
- Because increase in rotor resistance, efficiency becomes less and poor speed regulations.
- It is possible in slip wound rotor only as rotor resistance cannot be changed in squirrel cage winding.

**4. Speed control by Cascade Arrangement:**

The two motors may be mechanically coupled together to drive a common load. Generally, the rotor output of first machine is connected to the stator of second machine in such a way that the revolving field of both the machines are in the same direction.

Under this condition, the resulting speed of field will be

$$N_{s_1} = \left(\frac{120\ f}{P_1 + P_2}\right) \text{ (When the machines are cumulatively coupled)}$$

$$N_{s_2} = \left(\frac{120\ f}{P_1 - P_2}\right) \text{ (When the machines are differentially coupled)}$$

Where, f = supply frequency

$P_1$ and $P_2$ = number of poles of machines I and II, respectively.

**5. Slip power Recovery:**

In this method, external voltage is injected into the rotor through the slip ring at slip frequency.

As,

$$T_e \propto sV_2^2$$

(i) Injecting the voltage to add the existing rotor voltage which will reduce slip to keep torque constant.

(ii) Injecting the voltage to oppose the existing rotor voltage which will increase slip to keep torque constant.

## Synchronous Machines

A three-phase synchronous machine is a doubly excited ac machine because its field winding is energized from a DC source and its armature winding is connected to an AC source. It rotates with speed of revolving field i.e. synchronous speed.

Synchronous machines are divided in two parts based on their rotor construction:

1. Cylindrical Rotor Machines
2. Salient pole Rotor Machines

- The synchronous speed $N_s$ and synchronous angular speed of a machine with p-pole pairs running on a supply of frequency $f_s$ are:

  $\omega_s = \frac{2\pi f_s}{p}$

- Slip $S = \frac{N_S - N}{N_S}$

  Where $N_S = \frac{120f}{p}$ = synchronous speed

### EMF EQUATION OF SYNCHRONOUS GENERATOR OR ALTERNATOR

The emf equation of Synchronous Generator or Alternator is given as:

Let,

Φ = Flux per pole, in Wb

P = Number of poles

$N_{ph}$ = Turns per phase

f = Frequency of induced emf in Hz

Z = Total number of conductors

$Z_{ph}$ = Conductors per phase connected in series = Z/3 as number of phases = 3

Consider a single conductor placed in a slot.

$$E_{ph} = \sqrt{2}\pi f \phi N_{ph} \text{ volts}$$

This is the general emf equation for an induced emf per phase for full pitch, concentrated type of winding.

**1. Pitch Factor or Coil Span Factor ($K_c$)**

It is defined as the ratio of resultant emf when the coil is short pitch to the result emf when the coil is full pitched. It is always less than one.

$$K_c = \frac{E_R \text{ when coil is short pitched}}{E_R \text{ when coil is full pitched}} = \frac{2E\cos\left(\frac{\alpha}{2}\right)}{2E}$$

$$K_c = \cos\left(\frac{\alpha}{2}\right)$$

- Chording angle to eliminate $n^{th}$ harmonics $(\alpha) = \frac{180°}{n}$.
- Coil span to eliminate $n^{th}$ harmonics, $(\beta) = 180\left(\frac{n-1}{n}\right)$.

**2. Distribution Factor ($K_d$)**

The distribution factor is defined as the ratio of the resultant emf when coils are distributed to the resultant emf when coils are concentrated. It is always less than one.

$$K_d = \frac{E_R \text{ when coils are distributed}}{E_R \text{ when coils are concentrated}} = \frac{2E\sin\left(\frac{m\beta}{2}\right)}{2mE\sin\left(\frac{\beta}{2}\right)}$$

$$K_d = \frac{\sin\left(\frac{m\beta}{2}\right)}{m\sin\left(\frac{\beta}{2}\right)}$$

Where, m = Slots per pole per phase
β = Slot angle = 180°/n
n = Slots per pole

When β is very small and m is large then the total phase spread is (mβ). So, generalised expression for the derivation of emf equation of Synchronous generator or Alternator can be written as

$E_{ph} = \sqrt{2}\pi K_c K_d f \Phi N_{ph}$ Volts

For full pitch coil, $K_c = 1$

For concentrated winding $K_d = 1$

**Note:**

- **Coil span** is the distance between two sides of the coil. It is expressed in terms of degrees, pole pitch, no. of slots/pole etc.
- **Pole pitch** is the distance between two identical points on two adjacent poles.
  Pole pitch is always 180° electrical = slots/pole.
- $\theta_{elec} = \frac{P}{2}\theta_{mech}$
- **Slot pitch or slot angle:** (β) Slot angle is the angle for each slot.
- Speed of space harmonics of order $(6k \pm 1)$ is $\frac{1}{(6k \pm 1)}N_s$

  Where, $N_s$ = Synchronous speed = $\frac{120f}{p}$
- The order of slot harmonics is $\left(\frac{2s}{p} \pm 1\right)$

  where s = No. of slots, P = No. of poles

- Slot harmonics can be eliminated by skewing the armature slots and fractional slot winding. The angle of skew = $\theta_s$ = β (slot angle) = 2(harmonic pole pitches) = 1 slot pitch.
- Distribution factor for slot harmonics,

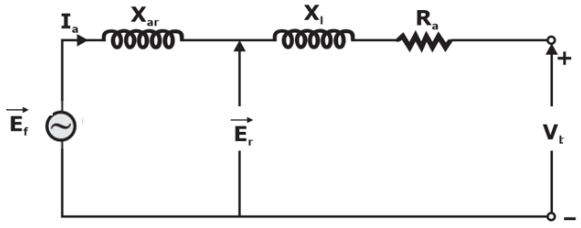
  $k_d\left(\frac{2s}{p}\pm 1\right)$

  Is $k_{d_1} = \frac{\sin\frac{my}{2}}{m\sin\frac{y}{2}}$ i.e., same as that of fundamental one.

- Pitch factor for slot harmonics, $k_p\left(\frac{2s}{p}\pm 1\right) = k_{p_1} = \cos\frac{\propto}{2}$

**Synchronous Generator:**

For a synchronous generator with stator induced voltage $E_s$, stator current $I_s$ and synchronous impedance $Z_s$, the terminal voltage V is:




**Equivalent circuit of synchronous generator**

$$\vec{V}_t = \vec{E}_f\angle\delta - \vec{I}_a(R_a + jX_s)$$

Where, $R_s$ is the stator resistance and $X_s$ is the synchronous reactance.

OR

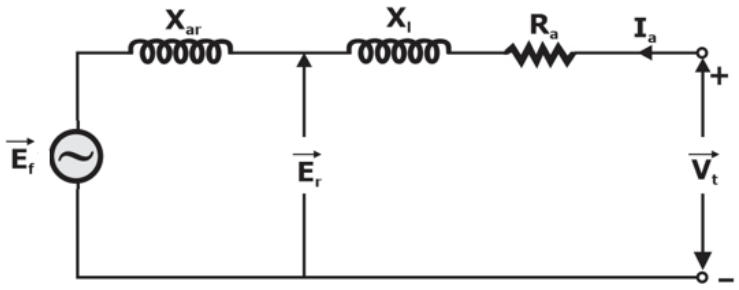
$$E_f = \sqrt{(V\cos\phi + I_aR_a)^2 + (V\sin\phi \pm I_aX_s)^2}$$

$+ \Rightarrow$ lag p.f.

$- \Rightarrow$ leading p.f.

**Synchronous Motor:**

For a synchronous motor with stator induced voltage $E_s$, stator current $I_s$ and synchronous impedance $Z_s$, the terminal voltage V is:




**Equivalent circuit of synchronous motor**

$$\vec{V}_t = \vec{E}_f \angle -\delta + \vec{I}_a(R_a + jx_s)$$

Where, $X_s$ is synchronous reactance.

$$E = \sqrt{\left(V\cos\phi - I_aR_a\right)^2 + \left(V\sin\phi \mp I_aX_s\right)^2}$$

$- \Rightarrow$ lag p.f.

$+ \Rightarrow$ leading p.f.

where $R_s$ is the stator resistance and $X_s$ is the synchronous reactance

**Voltage Regulation:**

- % regulation = $\frac{|E| - |V|}{|V|} \times 100$

  $E - V = I_aZ_s$

  $\therefore$ % regulation = $\frac{E - V}{V} = \frac{I_aZ_s}{V} \times 100$

  $\therefore$ regulation $\propto Z_s$

  $\therefore$ As $Z_s$ increases, voltage regulation increases.

- Condition for zero/minimum voltage regulation is, $\cos(\theta + \varphi) = -\frac{I_aZ_s}{2V}$

- Condition for max. Voltage regulation is, $\varphi = \theta$

**Short circuit ratio (SCR)**

It is defined as the ratio of field current required to produce rated voltage on open circuit to the field current required to produce rated armature current.

As SCC is linear,

$$SCR = \frac{I_{SC}}{I_a(\text{rated})} = \frac{\left(V_t(\text{rated})/\sqrt{3}\right)}{X_s(\text{saturated})} = \frac{Z_{base}}{X_s(\text{saturated})}$$

$$SCR = \frac{1}{X_s(pu)}$$

$$SCR \propto \frac{1}{X_a} \propto \frac{1}{\text{Armature reaction}}$$

Voltage regulation $\propto$ Armature reaction

$$\therefore \ SCR \propto \frac{1}{\text{Voltage regulation}}$$

$\therefore$ Small value of SCR represents poor regulation.

$$\phi_a = \frac{\text{Armature mmf}}{\text{Reluctance}}$$

But reluctance $\propto$ Air gap

$$\therefore \phi_a = \frac{\text{Armature mmf}}{\text{Airgap}}$$

$$\phi_a \propto \frac{1}{\text{Airgap length}}$$

$$\text{Armature reaction} \propto \phi_a \propto \frac{1}{\text{Airgap length}}$$

$$\therefore \text{SCR} \propto \frac{1}{\text{Armature reaction}} \propto \text{Airgap length}$$

Air gap length $\propto$ SCR

$\therefore$ Machine size $\propto$ SCR, Cost $\propto$ SCR

$$\text{Power} = \frac{EV}{X_a}\sin\delta \Rightarrow P \propto \frac{1}{X_s} \propto \text{SCR}$$

Power $\propto$ SCR

$\therefore$ Large value of SCR represents more power output.

## POWER FLOW IN CYLINDRICAL ROTOR SYNCHRONOUS MACHINE

### Power flow in generator:

$$S_{out} = \vec{V}_t \vec{I}_a^{\,*}$$

$$S_{out} = P_{out} + jQ_{out}$$

$$P_{out} = \frac{E_f V_t}{Z_s}\cos(\theta - \delta) - \frac{V_t^2}{Z_s}\cos\theta$$

$$Q_{out} = \frac{E_f V_t}{Z_s}\sin(\theta - \delta) - \frac{V_t^2}{Z_s}\sin\theta$$

$P_{out}$ will be maximum when $\theta = \delta$

$$\boxed{P_{out(max)} = \frac{E_f V_t}{Z_s} - \frac{V_t^2}{Z_s}\cos\theta}$$

# Neglecting armature resistance, $R_a$

Then, $Z_s = X_s$ and $\theta = 90°$

By putting the value in power equation,

$$\boxed{P = \frac{V_t E_f}{X_s}\sin\delta}$$

Reactive power:

$$\boxed{Q_{out} = \frac{V_t}{X_s}(E_f\cos\delta - V_t)}$$

**Power Flow in motor:**

$$S_{input} = P_{input} + jQ_{input} = V_t I_a^*$$

$$\therefore P_{in} = \frac{V_t^2}{Z_s}\cos\theta - \frac{V_t E_f}{Z_s}\cos(\theta+\delta)$$

$$\therefore Q_{in} = \frac{V_t^2}{Z_s}\sin\theta - \frac{V_t E_f}{Z_s}\sin(\theta+\delta)$$

# Neglecting armature resistance, $R_a$

Then, $Z_s = X_s$ and $\theta = 90°$

By putting the value in power equation,

$$\boxed{P = \frac{V_t E_f}{x_s}\sin\delta}$$

Reactive power:

$$\boxed{Q_{out} = \frac{V_t}{x_s}(V_t - E_f\cos\delta)}$$

**Conclusion:**

**For generator:**

Case 1: When $E_f\cos\delta = V_t$ , i.e. normally excited, then $Q_{out} = 0$ and therefore operating at unity factor.

Case 2: When $E_f\cos\delta > V_t$ i.e. overexcited, Qout = +ve, supplying lagging VARS and therefore operating at lagging p.f.

Case 3: When $E_f\cos\delta < V_t$ i.e. under excited, Qout = –ve, supplying leading VARs and therefore operating at leading p.f.

**For motor:**

Case 1: When $E_f\cos\delta = V_t$ i.e. normally excited motor than $Q_{in} = 0$ and therefore operating at unity power factor.

Case 2: When $E_f\cos\delta > V$ i.e. overexcited motor than $Q_{in}$ = –ve i.e. taking leading VARs and therefore operating at leading p.f.

Case 3: When $E_f \cos\delta < V$ i.e. under excited motor, Qin = +ve, i.e. taking lagging VARs and operating at lagging p.f.

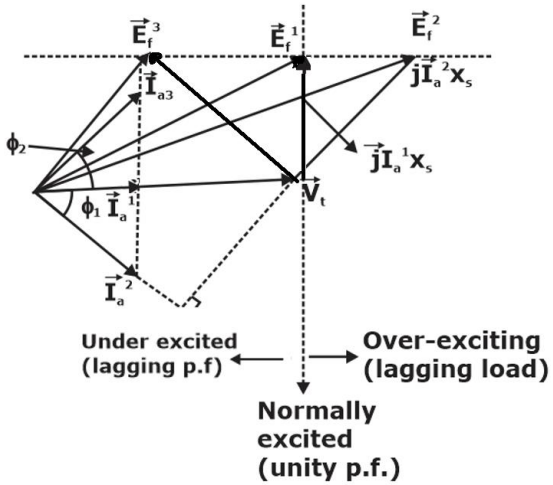
## EFFECT OF CHANGE IN EXCITATION AT CONSTANT (KW) OUTPUT

Power can be expressed as:

$$P = \frac{VE_f}{X_s}\sin\delta \qquad P = VI_a\cos\phi$$

$$P \propto E_f \sin\delta \qquad P \propto I_a\cos\phi$$

**A) Generator:**

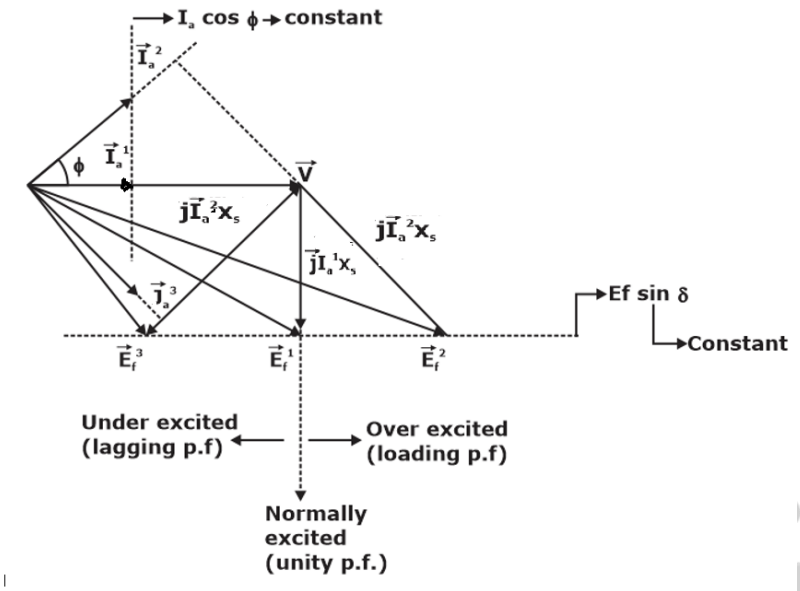



**Effect of change in excitation on alternator**

**Conclusion:**

1. For unity power factor load ($I_a^1$), generator is normally excited ($E_f^1$)
2. For lagging power factor load ($I_a^2$), generator is over-excited ($E_f^2$)
3. For leading power factor load ($I_a^3$), generator is under-excited ($E_f^3$)

**B) Motor:**



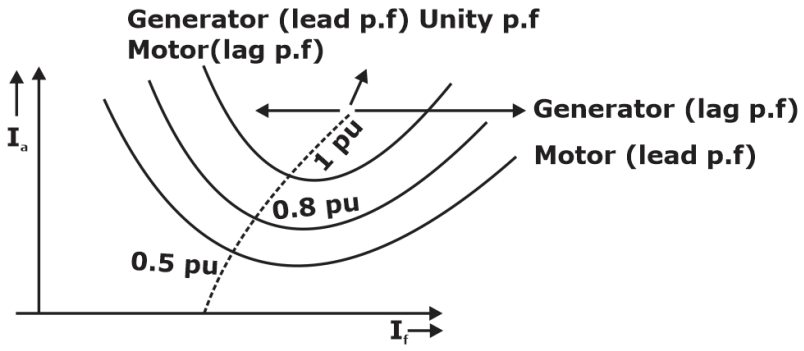

**Effect of change in excitation on synchronous motor**

**Conclusion:**

1. For unity power factor load ($I_a^1$), motor is normally excited ($E_f^1$)
2. For leading power factor load ($I_a^2$), motor is over-excited ($E_f^2$)
3. For lagging power factor load ($I_a^3$), motor is under-excited ($E_f^3$)

**V curve:**

It is the plot of armature current ($I_a$) versus field current ($I_f$) at constant load.

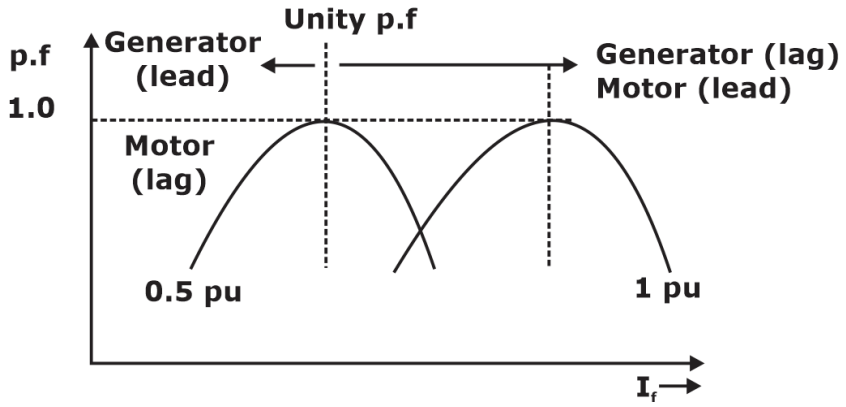



**V-curves**

**Note:** Armature current is minimum at the normally excited machine at unity p.f load.

**Inverted 'V' curve:**

These are the curves plotted between power factor (p.f) and field current ($I_f$).




**Inverted V-curves**

**Note:** p.f is unity at normally excited machine at unity p.f. load.

### Synchronizing power coefficient or stability factor

Synchronizing power coefficient or stability factor $P_{sy}$ is given as:

$$P_{sy} = \frac{dp}{d\delta} = \frac{d}{d\delta}\left(\frac{EV}{X_s}\sin\delta\right) = \frac{EV}{X_s}\cos\delta$$

$P_{sy}$ is a measure of stability.

$\therefore$ Stability $\propto P_{sy}$

But, $P_{sy} \propto \frac{1}{X_s} \propto SCR$

$\therefore$ Stability $\propto$ SCR

Stability $\propto$ SCR $\propto$ air gap length

$\therefore$ Stability $\propto$ air gap length

$\therefore$ Synchronizing Power coefficient, $S_P$:

$$\boxed{S_P = \frac{\Delta P}{\Delta\delta} = \frac{VE_f}{X_s}\cos\delta \text{ Watts / Electric radian}}$$

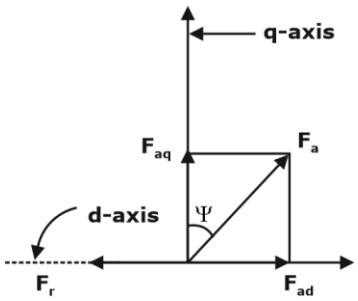
$S_p = P_{max} \text{ Cos } \delta$

Synchronizing power increases with decrease in load angle. This is the reason to synchronizing generator at no-load.

## SALIENT POLE SYNCHRONOUS MACHINE

### Two reaction theory

According to two reaction theory, the sinusoidal armature MMF, $F_a$ is resolved into two sinusoidal components:



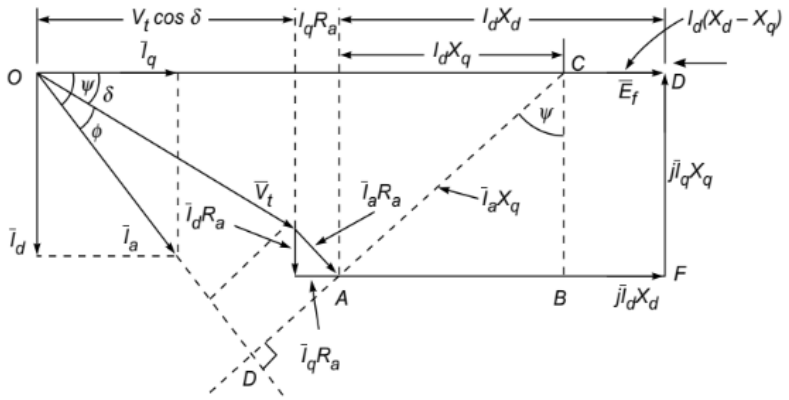

**Phasor diagram explaining two-reaction theory**

- $F_{ad}$ along d-axis
- $F_{aq}$, along q-axis

Where, $F_d = F_a \sin \varphi$

$F_q = F_a \cos \varphi$

The d-axis $F_{ad}$ component is magnetizing or demagnetizing depending upon, Ψ (angle of lead or lag). The q-axis components $F_{aq}$ produces only distortion of the field flux wave. $F_{ad}$ and $F_{aq}$ are in time quadrature with $E_f$, (produced by $I_d$ and $I_a$).




**Phasor diagram of salient pole synchronous generator at lagging load**

where,

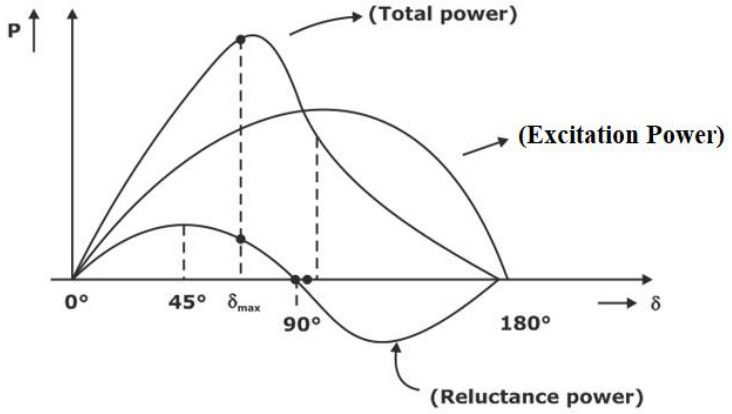
$\overrightarrow{E_f} = \overrightarrow{V_t} + \overrightarrow{I_a}R_a + j\overrightarrow{I_d}X_q + j\overrightarrow{I_q}X_q$

d-axis current, $I_d = I_a \sin \emptyset$

q-axis current, $I_q = I_a \cos \emptyset$

**Power flow in salient pole synchronous generator**

$$P = \underbrace{\frac{V E_f \sin\delta}{X_d}}_{\text{Excitation power}} + \underbrace{\frac{V^2}{2}\left(\frac{1}{X_q} - \frac{1}{X_d}\right)}_{\substack{\text{power due to saliency} \\ \text{or} \\ \text{reluctance power.}}} \sin 2\delta$$




**Power characteristics of salient pole synchronous generator**

**Note:**

- Cylindrical rotor Synchronous machine,

  The per phase power delivered to the infinite bus is given by $P = \frac{E_f V_t}{X_s} \sin\delta$

- Salient pole synchronous machine,

  The per phase power delivered to the infinite bus is given by

  $$P = \frac{E_f V_t}{X_d} \sin\delta + \frac{V_t^2}{2}\left[\frac{1}{X_q} - \frac{1}{X_d}\right] \sin 2\delta$$

**Condition for max. Power:**

- For cylindrical rotor machine:

  At constant $V_t$ and $E_f$, the condition for max. power is obtained by putting $\frac{dp}{d\delta} = 0$

  $$\therefore \frac{dp}{d\delta} = \frac{E_f V_t}{X_s} \cos\delta = 0$$

  $\cos\delta = 0$

  $\delta = 90°$

  Hence maximum power occurs at $\delta = 90°$

- For salient – pole synchronous machine:

  $$\frac{dp}{d\delta} = 0$$

  $$\Rightarrow \frac{V_t E_f}{X_d} \cos\delta + V_t^2 \left(\frac{1}{x_q} - \frac{1}{x_d}\right) \cos 2\delta = 0$$

øthan 90°.

$\therefore$ Maximum power occurs at $\delta < 90°$

- Synchronizing power = $P_{sy}.\Delta\delta. = \frac{EV}{X_s}\cos\delta\Delta\delta$
- Synchronizing torque = $\frac{\text{Synchronizing power}}{\omega}$

**SLIP TEST**

- When the stator MMF is aligned with the d – axis of field poles then flux $\emptyset_d$ per pole is set up and effective reactance offered by the alternator is $X_d$.

  $X_d = \frac{\text{Maximum voltage}}{\text{Minimum current}} = \frac{(V_t)\text{line}(\text{at min}, I_a)}{\sqrt{3}I_a(\text{min})}$ = Direct axis reactance

- When the stator MMF is aligned with the q – axis of field poles then flux $\emptyset_q$ per pole is set up and the effective reactance offered by the alternator is $X_q$.

  $X_q = \frac{\text{Minimum voltage}}{\text{Maximum current}} = \frac{(V_t)\text{line}(\text{at max}, I_a)}{\sqrt{3}I_a(\text{max})}$ = Quadrature axis reactance

**Hunting**

- A synchronous machine operates satisfactorily if the mechanical speed of the rotor is equal to the stator field speed i.e., if relative speed between rotor and stator fields is equal to zero. any departure from these conditions gives rise to synchronizing forces, which tend to maintain this equality.
- The phenomena of oscillation of rotor about its final equilibrium position is called hunting. The orientation of phasor $E_f$ changes relative to $V_t$ and because of this reason, hunting is called phase swinging.

**METHODS OF STARTING OF SYNCHRONOUS MOTOR**

- Motor starting with an external prime Mover
- Damper winding

****

BYJU'S EXAM PREP

# Electrical Engineering

## Digital Circuits

SHORT NOTES

# IMPORTANT FORMULAS ON DIGITAL CIRCUITS

## Number System and Codes

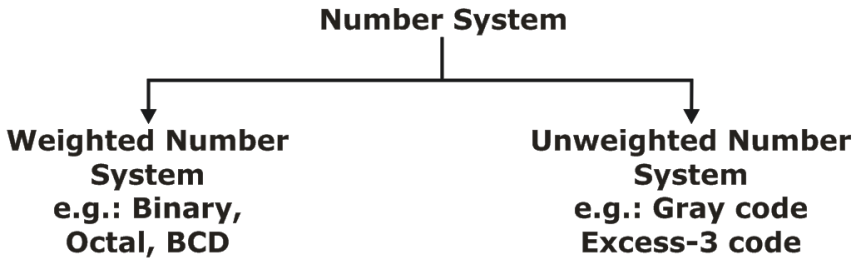



**Fig. 1**

**Types of Number System:**

The number can be represented in various ways to show the data and process it on the processing devices.

| Decimal Number System | Hexadecimal Number System | Octal Number System | Binary Number System |
|---|---|---|---|
| 0 | 0 | 0 | 0000 |
| 1 | 1 | 1 | 0001 |
| 2 | 2 | 2 | 0010 |
| 3 | 3 | 3 | 0011 |
| 4 | 4 | 4 | 0100 |
| 5 | 5 | 5 | 0101 |
| 6 | 6 | 6 | 0110 |
| 7 | 7 | 7 | 0111 |
| 8 | 8 | 10 | 1000 |
| 9 | 9 | 11 | 1001 |
| 10 | A | 12 | 1010 |
| 11 | B | 13 | 1011 |
| 12 | C | 14 | 1100 |
| 13 | D | 15 | 1101 |
| 14 | E | 16 | 1110 |
| 15 | F | 17 | 1111 |

**Table 1: Counting in different number system**

A number system with base 'r', contains 'r' different digits and they are from 0 to r –1.

**Decimal to other codes conversions:**

To convert decimal number into other system with base 'r', divide integer part by r and multiply fractional part with r.

**Other codes to Decimal Conversions:**

$(x_2x_1x_0.y_1y_2)_r \rightarrow (A)_{10}$ , $A = x_2r^2 + x_1r + x_0 + y_1r^{-1} + y_2r^{-2}$

**Hexadecimal to Binary:**

Convert each Hexadecimal digit into 4 bits binary.

$(5AF)_{16} \rightarrow \underset{5}{(0101}\ \underset{A}{1010}\ \underset{F}{1111)_2}$

**Binary to Hexadecimal:**

Grouping of 4 bits into one hex digit.

$(110101.11)_2 \rightarrow 00110101.1100 \rightarrow (35.C)_{16}$

**Octal Binary and Binary to Octal:**

Same procedure as discussed above but here group of 3 bits is made.

**Codes:**

**Binary coded decimal (BCD):**

- In BCD code each decimal digit is represented with 4 bit binary format.

  $\text{Eg}: (943)_{10}\left(\underset{9}{1001}\ \underbrace{\underset{4}{0100}}\ \underset{3}{0011}\right)_{BCD}$

- It is also known as 8421 code.
- Invalid BCD codes are the codes whose decimal equivalent is more than 9. i.e. Valid BCD codes ranges from 0 to 9.

  Total number of codes possible $\rightarrow 2^4 \Rightarrow 16$

  Valid BCD codes $\rightarrow 10$

  Invalid BCD codes $16 - 10 \Rightarrow 6$

  There 1010, 1011, 1100, 1110 and 1111

**Excess-3 codes: (BCD + 0011)**

- It can be derived from BCD by adding '3' to each coded number.
- It is unweighted and self-complementing code.

**Gray Code:**

It is also known as minimum change codes or unit distance code or reflected code.

**Binary code to Gray code:**

In order to find Gray code from Binary code, XOR Gate is applied between the present binary bit and the next binary bit starting from the MSB side(keeping MSB of Binary and Gray as same) e.g.

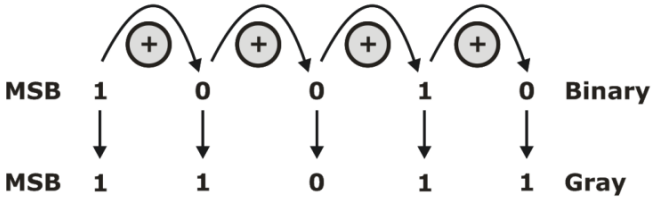



**Fig. 2**

**Gray code to Binary code:**

In order to find Binary code from Gray code, XOR Gate is applied between the present binary bit and the next gray bit starting from the MSB side (keeping MSB of Gray and Binary as same) e.g.

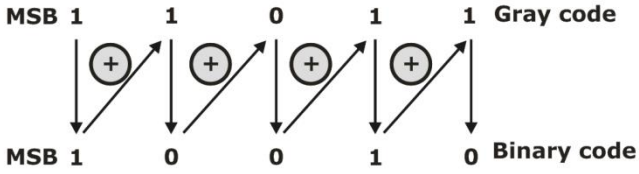



**Fig. 3**

**Alpha Numeric code:**

EBCDIC (Extended BCD interchange code)

It is an 8-bit code. It can represent 128 possible characters.

- Parity method is most widely used schemes for error detection.
- Hamming code is most useful error correcting code.
- BCD code is used in calculators, counters.

**Complements:**

Its base is r then we can have two complements.

(i) (r – 1)'s complement

(ii) r's complement

To determine(r – 1)'s complement: First write maximum possible number in the given system and subtract the given number.

To determine r's complements: (r – 1)'s complement + 1

i.e. First write (r – 1)'s complement and then add 1 to LSB

**Example:**

Q. Find 7's and 8's complement of 2456

Sol. 7's Complement

$$\begin{array}{r} 7777 \\ -2456 \\ \hline 5321 \end{array}$$

8's Complement

$$\begin{array}{r} 5321 \\ +1 \\ \hline 5322 \end{array}$$

Q. Find 2's complement of 101.110

Sol. 1's complements 010.001

For 2's complement add 1 to the LSB

2's complement

$$\begin{array}{r} 010.001 \\ +1 \\ \hline 010.010 \end{array}$$

**Data Representation:**

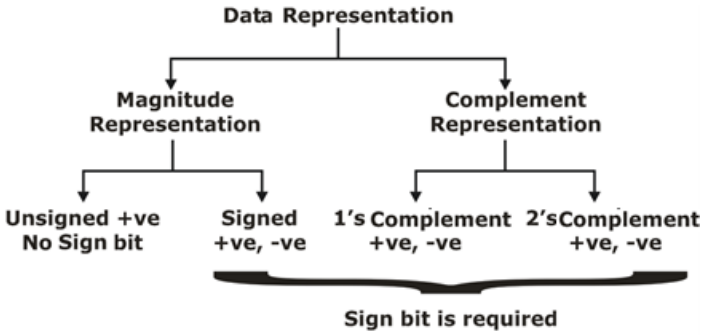



**Fig. 4**

**Unsigned Magnitude:**

Range with n bit

$\rightarrow$ 0 to $2^{n-1}$ $\quad +5 \Rightarrow 101$

$-5 \Rightarrow$ Not possible

**Signed Magnitude:**

Range with n bit

$-6 \Rightarrow \underbrace{1\quad 110}_{\text{sign bit with 4 bits}} \quad \underbrace{1\quad 0000110}_{\text{sign bit with 8 bits}}$

**1's complements:**

Range with n bit

$\rightarrow -\left(2^{n-1}-1\right)$ to $+\left(2^{n-1}-1\right)$

$+6 \Rightarrow 0110 \quad -6 \Rightarrow 1001$

**2's complements:**

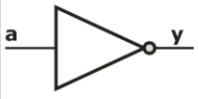
With n bit range $-2^{n-1}$ to $(2^{n-1}-1)$

$+6 \Rightarrow 0110 \quad -6 \Rightarrow 1010$

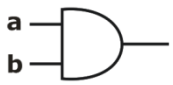
In any representation +ve numbers are represented similarly to +ve number in sign magnitude.

## Logic Gates

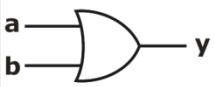
**NOT Gate:**

| Symbol | Truth-table | Boolean |
|---|---|---|
|   | a \| y<br>0 \| 1<br>1 \| 0 | y = a |

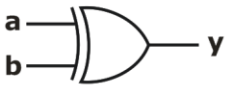
**AND Gate:**

| Symbol | Truth-table | Boolean |
|---|---|---|
|  | a b \| y<br>0 0 \| 0<br>0 1 \| 0<br>1 0 \| 0<br>1 1 \| 1 | y = ab |

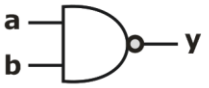
**OR Gate:**

| Symbol | Truth-table | Boolean |
|---|---|---|
|  | a b \| y<br>0 0 \| 0<br>0 1 \| 1<br>1 0 \| 1<br>1 1 \| 1 | y = a + b |

**Exclusive OR Gate:**

| Symbol | Truth-table | Boolean |
|---|---|---|
|  | a b \| y<br>0 0 \| 0<br>0 1 \| 1<br>1 0 \| 1<br>1 1 \| 0 | $y = a \oplus b$ |

**NAND Gate:**

| Symbol | Truth-table | Boolean |
|---|---|---|
|  | a b \| y<br>0 0 \| 1<br>0 1 \| 1<br>1 0 \| 1<br>1 1 \| 0 | $y = \overline{ab}$ |

**NOR Gate:**

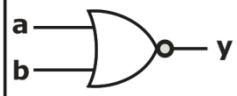
| Symbol | Truth-table | Boolean |
|---|---|---|
|  | a b \| y<br>0 0 \| 1<br>0 1 \| 0<br>1 0 \| 0<br>1 1 \| 0 | $y = \overline{a + b}$ |

**Fig.5**

**Note:-** NAND and NOR are Universal Gates.

**XOR gate:**

Symbol of two input XOR gate

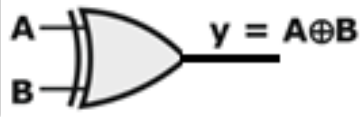



**Fig.6**

**X-NOR gate:**

Symbol for two input X-NOR gate

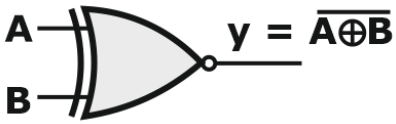



**Fig.7**

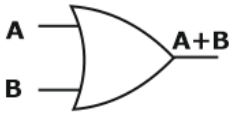
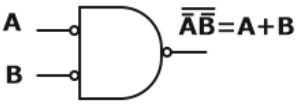
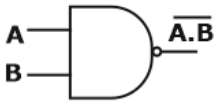
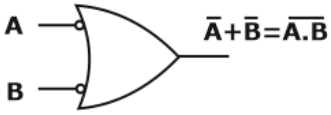
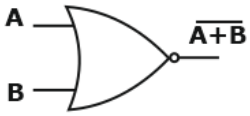
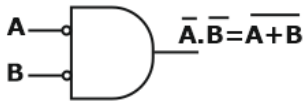
| Logic | Normal symbol | Alternate symbol |
|---|---|---|
| NOT | | |
| AND | | |
| OR |   |   |
| NAND |   |   |
| NOR |   |   |

**Table 2: Logic Gates representation**

## Minimization

**Boolean Laws:**

| | | |
|---|---|---|
| Commutative | $x \wedge y = y \wedge x$ | $x \vee y = y \vee x$ |
| Associative | $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ | $x \vee (y \vee z) = (x \vee y \vee z)$ |
| Distributive | $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ | $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ |
| Idempotence | $x \wedge x = x$ | $x \vee x = x$ |
| Absorption | $x \wedge (x \vee y) = x$ | $x \vee (x \wedge y) = x$ |
| Consensus theorem | $(x \wedge y) \vee (\bar{x} \wedge z) \vee (y \wedge z)$ $= (x \wedge y) \vee (\bar{x} \wedge z)$ | $(x \vee y) \wedge (\bar{x} \vee z) \wedge (y \vee z)$ $= (x \vee y) \wedge (\bar{x} \vee z)$ |
| Combining | $(x \wedge y) \vee (x \wedge \bar{y}) = x$ | $(x \vee y) \wedge (x \vee \bar{y}) = x$ |
| DeMorgan's | $\overline{(x \wedge y)} = \bar{x} \vee \bar{y}$ | $\overline{(x \vee y)} = \bar{x} \wedge \bar{y}$ |

**Table 3: Boolean law expressions**

**Dual of a function:**

The dual of logic function, f, is the function $f^D$ derived from f by substituting a $\wedge$ for each $\vee$, and $\vee$ for each $\wedge$. 1 for each 0, and 0 for each 1.

$$f(a,b) = (a \wedge b) \vee (b \wedge c) \qquad f^D(a,b) = (a \vee b) \wedge (b \vee c)$$

**Boolean Functions:**

Any Boolean expression can be expressed in two forms

- Sum of Product form (SOP)
- Product of Sum form (POS)

**SOP form:**

The SOP expression usually takes the forms of two or more variables ORed together.

$$Y = \bar{A}BC + A\bar{B} + AC$$

$$Y = A\bar{B} + B\bar{C}$$

SOP forms are used to write logical expression for the output becoming logic '1'.

**POS form:**

The POS expression usually takes the form of two or more OR variables within parentheses, ANDed with two or more such terms.

**Karnaugh Map (K-MAP):**

The K-map is a graphical method which provides a systematic method for simplifying the Boolean expressions. In n variable K-map, there are $2^n$ cells.

**Simplification Rules:**

1. Construct the K-map and place 1's in those cells corresponding to the 1's in the truth table. Place the 0's in the other cells.
2. Examine the map for adjacent 1's and loop those 1's which are not adjacent to any other 1's. These are called isolated 1's.
3. Next, look for those 1's which are adjacent to only one other 1. Loop any pair containing such a 1.
4. Loop any octet even if it contains some 1's that have already been looped.
5. Loop any quad that contains one or more 1's which have not already been looped, making sure to use the minimum number of loops.
6. Form the OR sum of all the terms generated by each loop.

**Implicants:**

Implicants is a product term on the given function, and for that combination, the function output must be 1.

**Prime Implicant (PI)**

Prime implicant is a smallest possible product term of the given function.

**Essential Prime Implicants (EPI)**

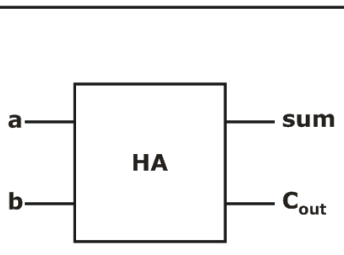
EPI is a prime implicant and it must cover at least one minterm, which is not covered by other PI.

**Combinational Circuits**

The combinational circuit has 'n' input variables and 'm' output variables. Since, the number of input variables is n, there are $2^n$ possible combinations of bits at the input.

**Adders:**

**Half Adder**




Logic circuit

| a | b | $C_{out}$ | sum |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 |

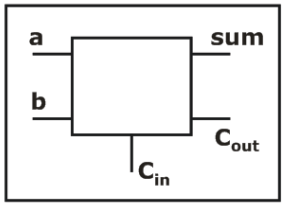
Truth table

$Sum = \bar{a}b + a\bar{b} = a \oplus b$

$C_{out} = ab$

Equations

**Full Adder**




Logic circuit

| $C_{in}$ | a | b | $C_{out}$ | sum |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

Truth table

$Sum = \bar{C}_{in} X + C_{in}\bar{X} = C_{in} \oplus X = C_{in} \oplus a \oplus b$

$C_{out} = C_{in}(a \oplus b) + ab$

Equations

**Fig. 8**

**Binary parallel adder:**

An n-bit parallel adder can be constructed using n number of full adders which are connected in parallel, and hence; it is also known as parallel adder such that the previous carry or carry input for adder 0 is set to zero. The carry output of each adder is connected to the carry input of the next higher order adder. Hence, it is also known as carry propagate adder.

**Carry look ahead adder:**

The look ahead carry adder speeds up the operation by eliminating this ripple carry delay. It examines all the input bits simultaneously, and also generates the carry in bits for all the stages simultaneously. The method of speeding up the addition process is based on the two additional functions of the full adder called the carry generate and carry propagate functions.

**Multiplexer:**

- $2^n$ inputs; 1 output and 'n' select lines.
- It can be used to implement Boolean function by selecting select lines as Boolean variables.
- For implementing 'n' variables Boolean function, $2^n \times 1$ MUX is enough.
- For implementing "n+1" variables Boolean function, $2^n \times 1$ MUX + NOT gate is required.

- For implementing "n+2" variables Boolean function, $2^n \times 1$ MUX + Combinational Circuit is required.
- If you want to design $2^m \times 1$ MUX using $2^n \times 1$ MUX. You need two $2^n \times 1$ MUXes.

**Demultiplexer:**

- 1 input; $2^n$ outputs and 'n' select lines.
- It can be used to implement Boolean function by selecting select lines as Boolean variables.
- For implementing 'n' variables Boolean function, $1 \times 2^n$ DeMUX is enough.
- For implementing "n+1" variables Boolean function, $1 \times 2^n$ DeMUX + NOT gate is required.
- For implementing "n+2" variables Boolean function $1 \times 2^n$ DeMUX + Combinational Circuit is required.
- If you want to design $1 \times 2^m$ DeMUX using $1 \times 2^n$ DeMUX. You need two $2^n \times 1$ DeMUXes.

**Comparison between Multiplexer and Demultiplexer:**

| S.No. | Parameter of comparison | Multiplexer | Demultiplexer |
|---|---|---|---|
| 1. | Type of logic circuit | Combinational | Combinational |
| 2. | Number of data inputs | M | 1 |
| 3. | Number of select inputs | N | N |
| 4. | Number of data output | 1 | M |
| 5. | Relation between input/output lines and select lines | $M = 2^N$ | $M = 2^N$ |
| 6. | Operation principle | Many to 1 or data selector | 1 to many or data distributor |

**Table 4: Comparison between Multiplexer and Demultiplexer**

**Decoder:**

- n input and $2^n$ output.
- Used to implement the Boolean function. It will generate required min terms at the output and those terms should be "OR" ed to get the results.
- Suppose it consists of more min terms then connect the max terms to NOR gate then it will give the same output with less no. of gates.
- For carry look ahead adder, delay = $2 \times t_{pd}$.

**Encoder:**

An encoder is a combinational logic circuit that performs the inverse operation of a decoder. An encoder has $2^n$ (or fewer) input lines and n output lines.

**Octal to Binary Encoder:**

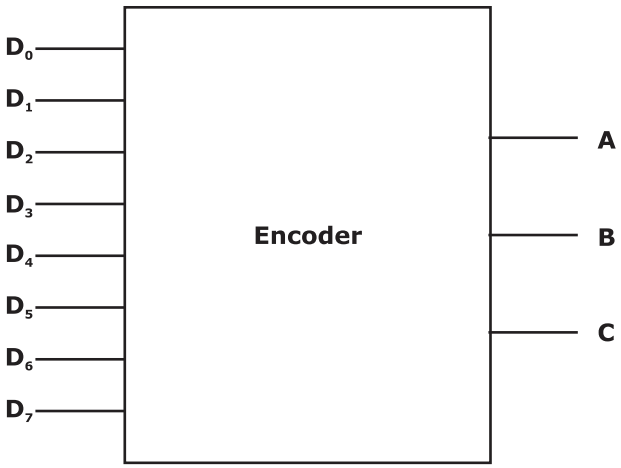



**Fig. 9**

The logical expression for the outputs are as follows:

$$A = D_4 + D_5 + D_6 + D_7$$

$$B = D_2 + D_3 + D_6 + D_7$$

$$C = D_1 + D_3 + D_5 + D_7$$

**Realization of a Octal to Binary Encoder using Logic Gates:**

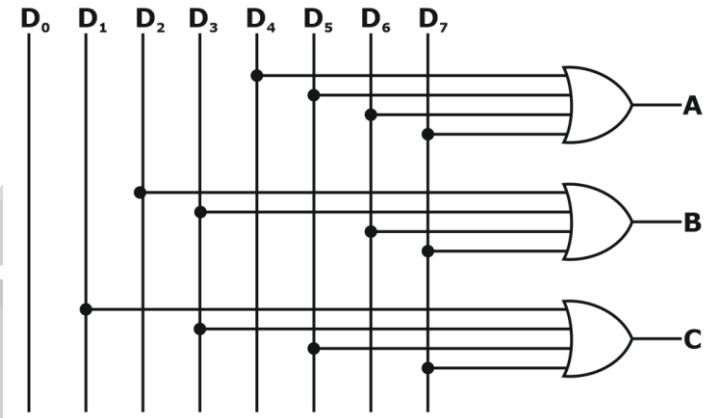



**Fig. 10**

**Subtractors:**

**Half subtractor:**

A half subtractor is a combinational logic circuit, which performs the subtraction of two 1-bit numbers. It subtracts one binary digit from another to produce a DIFFERENCE output & a BORROW output.

The truth table of half – subtractor, where A, B are the inputs, and difference (D) and borrow (B) are the outputs.

| Inputs | | Outputs | |
|---|---|---|---|
| **A** | **B** | **D** | **$B_{out}$** |
| 0 | 0 | 0 | 0 |

| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |

**Table 6: Truth table of Half – Subtractor.**

Difference, $D = \bar{A}B + A\bar{B} = A \oplus B$

Borrow, $B_{out} = \bar{A}B$

**Logic Diagram:**

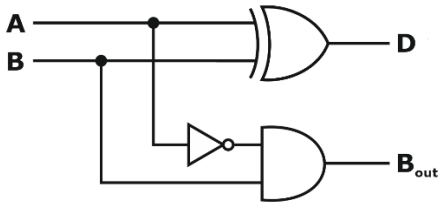



**Fig. 11**

**Full Subtractor:**

A full subtractor performs subtraction operation on two bits, a minuend and a subtrahend.

| Inputs | | | Outputs | |
|---|---|---|---|---|
| **A** | **B** | **$B_{in}$** | **D** | **$B_{out}$** |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

**Table 7: Truth table of Full subtractor.**

Difference, $D = A \oplus B \oplus B_{in}$

Borrow, $B_{in} = \bar{A}B + \bar{A}B_{in} + BB_{in}$

**Logic Diagram:**

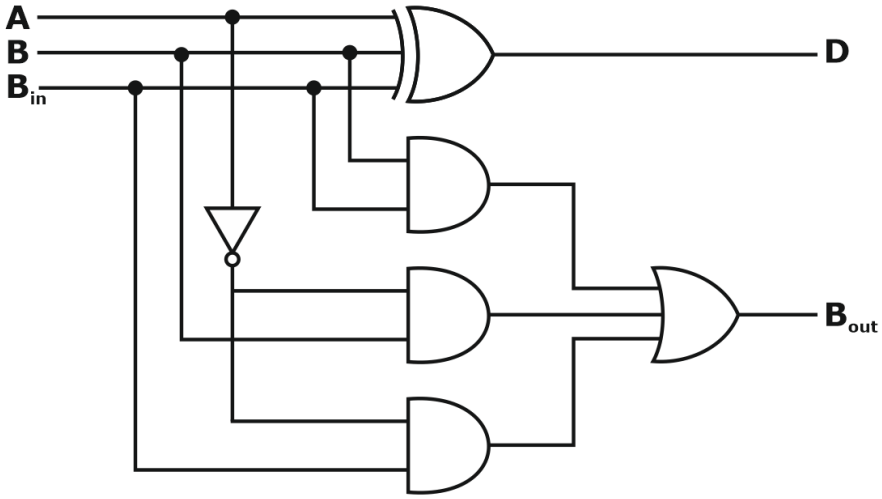



**Fig. 12**

**Comparator:**

The comparator has three outputs namely A>B, A=B and A<B. Depending upon the result of comparison, one of these outputs will go high.

**2-bit Magnitude Comparator:**

Let the two 2-bit numbers be $A = A_1A_0$ and $B = B_1B_0$

| Inputs | | | | Outputs | | |
|---|---|---|---|---|---|---|
| $A_1$ | $A_0$ | $B_1$ | $B_0$ | X(A<B) | Y(A=B) | Z(A > B) |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 |

**Table 9: Truth Table for a 2-bit Comparator.**

**Logic Diagram of 2-bit Comparator:**

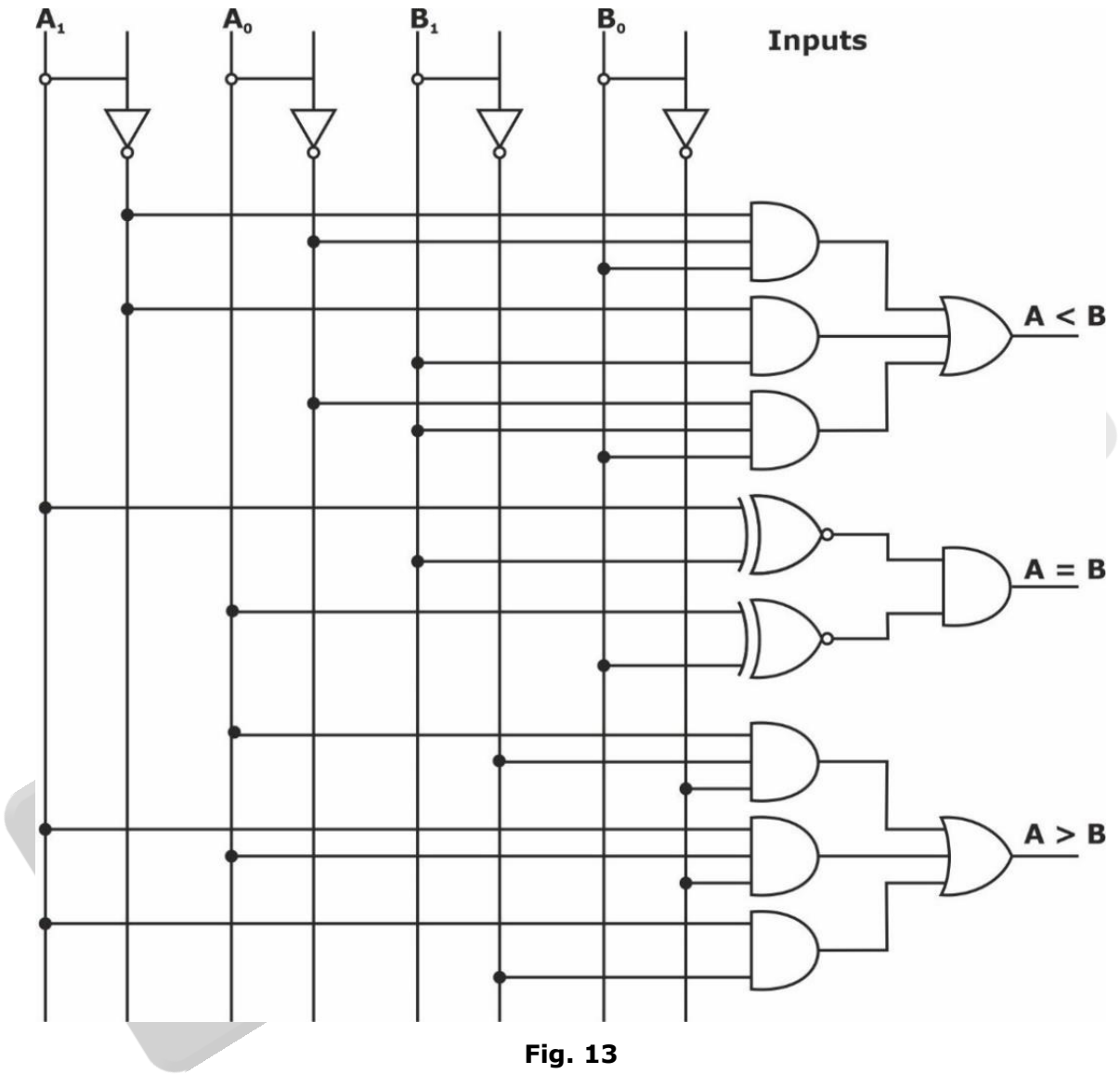



**Fig. 13**

**CODE CONVERTERS:**

A code converter is a combinational logic circuit which accepts the input information in one binary code, converts it and produces an output into another binary code.

**The truth table for 4-bit Binary and its Equivalent BCD:**

| Decimal | Binary Input | | | | BCD Output | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | A | B | C | D | $B_4$ | $B_3$ | $B_2$ | $B_1$ | $B_0$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 5 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 6 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 7 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 8 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 9 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 10 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 11 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| 12 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 13 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 14 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 15 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 |

**Table 11: Truth table for 4-bit Binary and its Equivalent BCD**

The minimized expression of outputs are as follows:

$B_1 = \bar{A}C + AB\bar{C}$

$B_2 = \bar{A}B + BC$

$B_3 = A\bar{B}\bar{C}$

$B_4 = AB + AC$

$B_0 = D$

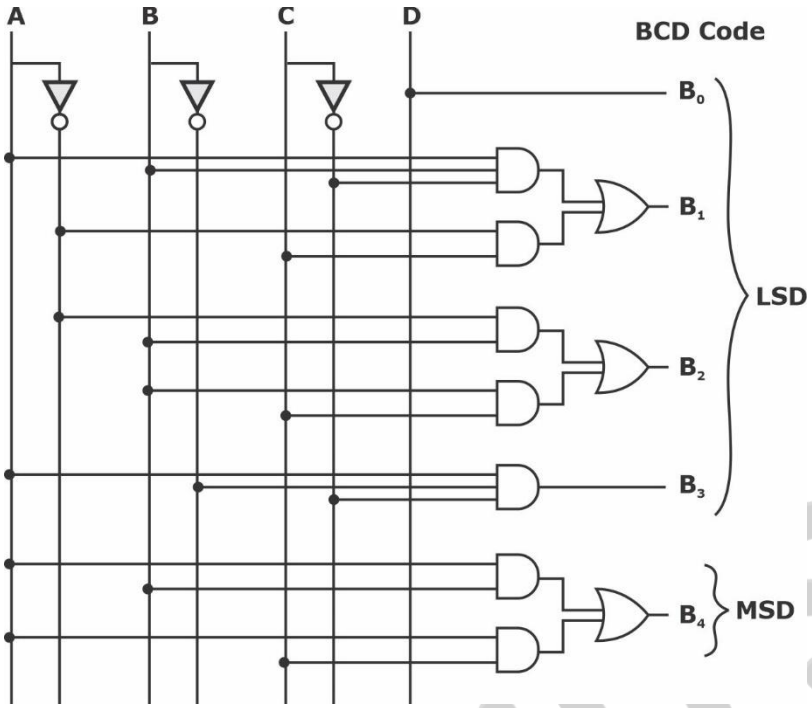



**Fig. 14**

**Parity Generator:**

Parity generators are circuits that accept an (n-1) bit data stream and generate an extra bit that is transmitted with the bit stream. This extra bit is referred to as the parity bit. The parity added in binary message is such that the total number of 1's in the message can be either odd or even according to the type of parity used.

**Even Parity Generator**:

The parity bit is '1' if there are odd number of 1's in the data stream and the parity bit is '0' if there are even number of 1's in the data stream.

The minimized expression for even parity generator is

$P = A \oplus B \oplus C \oplus D$

The logic diagram for the even parity generator is given as

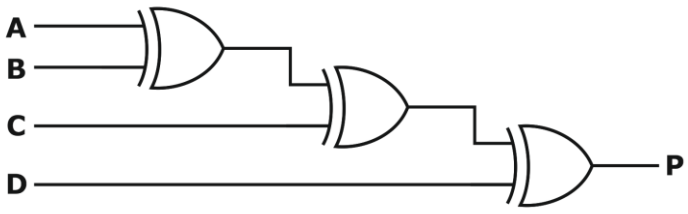



**Fig. 15**

**Odd Parity Generator:**

The parity bit is '0' for odd number of 1's and '1' for even number of 1's in the bit stream.

The minimized expression for odd parity generator is

$$P = (A \oplus C) \odot (B \oplus D)$$

The logic diagram of odd parity generator is given as

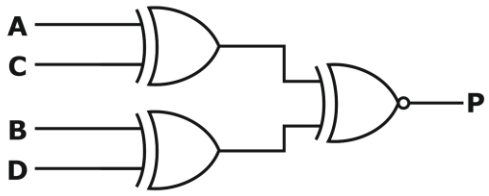



**Fig. 16**

**Sequential Circuits**

Sequential circuit include memory elements to store past data.

The flip-flop is a basic element of sequential logic circuits.

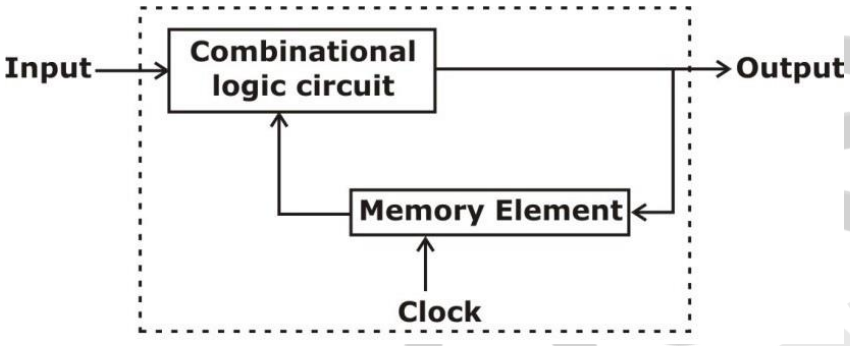



**Fig. 17**

There are two types of sequential circuits:

1. Synchronous Sequential Circuits
2. Asynchronous Sequential Circuits

**Difference between Synchronous and Asynchronous Sequential Circuits:**

| S.No. | Synchronous Sequential Circuits | Asynchronous Sequential Circuits |
|---|---|---|
| 1. | In synchronous circuits, the change in input signals can affect memory elements upon activation of clock signal. | In asynchronous circuits, change in input signals can affect memory elements at any instant of time. |
| 2. | In synchronous circuits, memory elements are clocked FF's. | In asynchronous circuits, memory elements are either unclocked FF's or time delay elements. |
| 3. | The maximum operating speed of the clock depends on time delays involved. | Since the clock is not present, asynchronous circuits can operate faster than synchronous circuits. |
| 4. | They are easier to design. | More difficult to design. |

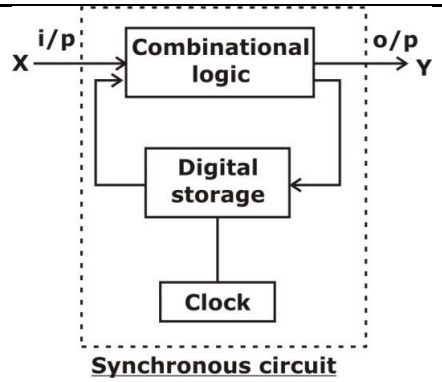
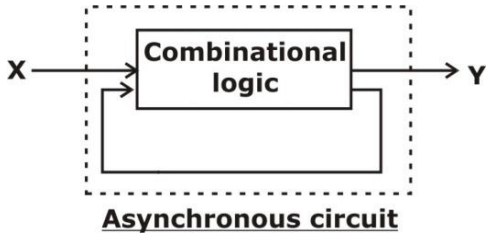
| | | |
|---|---|---|
| 5. |  |  |

**Table 12: Asynchronous and Synchronous Circuit**

**Difference between Latches and Flip-flops:**

| S.No. | Latch | Flip-flop |
|---|---|---|
| 1. | A latch is an electronic sequential logic circuit used to store information in an asynchronous arrangement. | A flip-flop is an electronic sequential logic circuit used to store information in a synchronous arrangement. It has two stable states and maintains its states for an indefinite period until a clock pulse is applied. |
| 2. | One latch can store one-bit information, but output state changes only in response to data input. | One flip-flop can store one-bit data, but output state changes with clock pulse only. |
| 3. | Latch is an asynchronous device and it has no clock input. | Flip-flop has clock input and its output is synchronised with clock pulse. |
| 4. | Latch holds a bit value and it remains constant until new inputs force it to change. | Flip-flop holds a bit value and it remains constant until a clock pulse is received. |
| 5. | Latches are level-sensitive, and the output tracks the input when the level is high. Therefore, as long as the level is logic level 1, the output can change if the input changes. | Flip-flops are edge sensitive. They can store the input only when there is either a rising or falling edge of the clock. |

**Table 13: Latches and Flip Flops**

**RS Latch:**

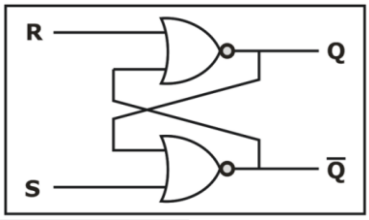



**Fig. 18**

| S R | Q $\overline{Q}$ | Comment |
|---|---|---|

| | | |
|---|---|---|
| 0 0 | Q $\overline{Q}$ | Hold |
| 0 1 | 0 1 | Reset |
| 1 0 | 1 0 | Set |
| 1 1 | | Illegal |

**Truth Table**

Where Q′ is the next state and Q is the current state.

**JK Flip Flop:**

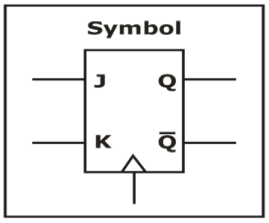



**Fig. 19**

| J K | Q′ $\overline{Q}$′ | Comment |
|---|---|---|
| 0 0 | Q $\overline{Q}$ | Hold |
| 0 1 | 0 1 | Reset |
| 1 0 | 1 0 | Set |
| 1 1 | $\overline{Q}$ Q | Toggle |

**Truth Table**

**T – Flip Flop:**

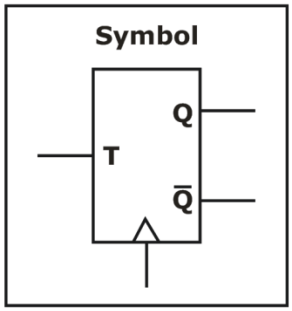
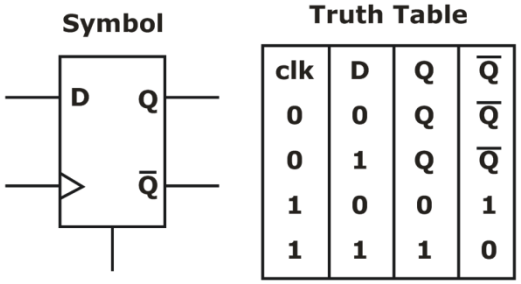



**Fig. 20**

| T | Q′ $\overline{Q}$′ | Comment |
|---|---|---|
| 0 | Q $\overline{Q}$ | Hold |
| 1 | $\overline{Q}$ Q | Reset |

**Truth Table**

**D – Flip Flop:**




| clk | D | Q | $\overline{Q}$ |
|---|---|---|---|
| 0 | 0 | Q | $\overline{Q}$ |
| 0 | 1 | Q | $\overline{Q}$ |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 |

**Fig. 21**

**Excitation tables:**

| QQ′ | S R | QQ′ | J K | QQ′ | D | QQ′ | T |
|---|---|---|---|---|---|---|---|
| 0 0 | 0 X | 0 0 | 0 X | 0 0 | 0 | 0 0 | 0 |
| 0 1 | 1 0 | 0 1 | 1 X | 0 1 | 1 | 0 1 | 1 |
| 1 0 | 0 1 | 1 0 | X 1 | 1 0 | 0 | 1 0 | 1 |
| 1 1 | X 0 | X 0 | X 0 | 1 1 | 1 | 1 1 | 0 |

- $Q(n+1) = S + R'Q = D = JQ' + K'Q = TQ' + T'Q$
- For ring counter total no. of states = n
- For twisted ring counter, total no. of states = "2n" (Johnson counter/switch tail ring counter).
- Master-slave circuit is used to counter the Race-around condition in JK flip flop. In Master slave Flip Flop, master is level triggered and slave is edge triggered.

**Propagation Delay Time:**

Propagation delay time is the time interval required after an input signal has been applied for the resulting output change to occur.

**Set-up time ($t_s$):**

It is the minimum time interval required for the logic levels (0 or 1) to be maintained constantly on the inputs (J, K or D) prior to the triggering edge of the clock pulse for the levels to be reliably clocked into the flip-flop.

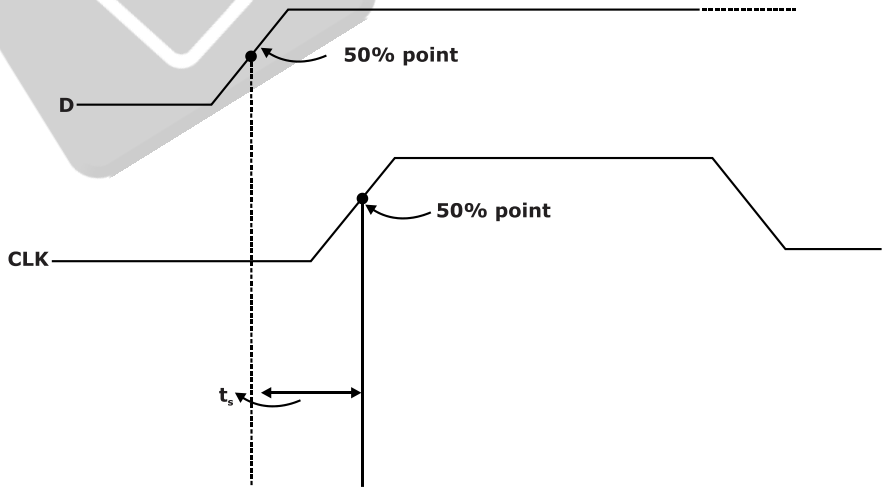



**Fig. 22**

**Hold time ($t_h$):**

It is the time for which the data must remain stable after the triggering edge of the clock.

**Clock-pulse width:**

The minimum time duration for which the clock pulse must remain HIGH and LOW which are designed by manufacturers. Failure to clock pulse width results in unreliable triggering.

**Maximum clock frequency:**

The maximum clock frequency ($f_{max}$) is the highest rate at which flip-flop can be reliably operated.

**Designing of one Flip Flop by other flip flop**

The steps for designing of one flip flop or new flip flop using existing or same existing flip flop.

**Step 1:** Write the characteristic table for the designed flip flop.

**Step 2:** Write the excitation table for the available flip flop.

**Step 3:** Write the logical expression.

**Step 4:** Minimize the logical expression.

**Step 5:** Circuit Implementation.

**Shift Registers:**

**SISO (serial-in, serial-out) Shift Register**

- It is the slowest shift register among all the shift registers.
- To store n-bits in a n-bit SISO register, then minimum "n" clock pulses are required.
- To retrieve n-bits from a n-bit SISO register, then minimum "(n-1)" clock pulses are required.

**SIPO (serial-in, parallel-out) Shift Register**

- To store n-bits in a n-bit SIPO register, minimum "n" clock pulses are required.
- To retrieve n-bits from a n-bit SIPO register, no clock pulse is required.

**PISO (parallel-in, serial-out) Shift Register**

- To store n-bit in a n-bit PISO register, a single clock pulse is required.
- To retrieve n-bit from n-bit PISO register, minimum "(n-1)" clock pulses are required.

**PIPO (parallel-in, parallel-out) Shift Register**

- To store n-bit in n-bit PIPO register, only a single clock pulse is required.
- To retrieve n-bits from n-bit PIPO register, no clock pulse is required.

**Universal shift register**

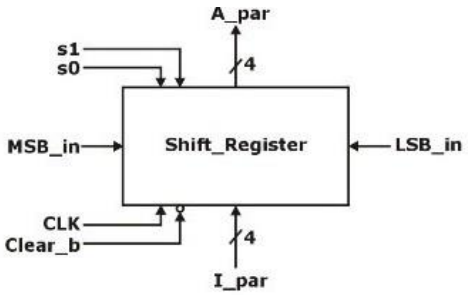



**Fig. 23**

The function for the Universal Shift Register is as follows:

| Mode Control | | Register Operation |
|---|---|---|
| $S_1$ | $S_0$ | |

| | | |
|---|---|---|
| 0 | 0 | No change |
| 0 | 1 | Shift right |
| 1 | 0 | Shift left |
| 1 | 1 | Parallel load |

**Table 14: Truth Table of Universal Shift Register**

**Applications of shift registers**

(a) Delay line: A shift register can be used to introduce a delay (Δt) in signals

$$\Delta t = N \times \frac{1}{f_c}$$

Where N is number of stages & $f_c$ is the clock frequency.

(b) Serial-to-parallel converter

(c) Parallel-to-serial converter

(d) Ring counter

(e) Twisted ring counter

(f) Sequence counter

**Counters:**

| S.No. | Asynchronous Counter | Synchronous Counter |
|---|---|---|
| 1. | Clock input is applied to LSB FF. The output of first FF is connected as clock to next FF. | Clock input is common to all FF. |
| 2. | All Flip-Flop are toggle FF. | Any FF can be used. |
| 3. | Speed depends on number of FF used for n bit. $f_{max} = 1/(n*t_n)$ | Speed is independent of number of FF used. $f_{max} = 1/t_p$ |
| 4. | No extra logic Gates are required. | Logic Gates required based on design. |
| 5. | Cost is less. | Cost is more. |

**Table 15: Difference between Asynchronous(ripple) and Synchronous Counter**

**ADC & DAC**

**Analog to Digital converter (ADC):**

- Resolution = $\frac{FSV}{2^n - 1}$ where FSV means Full Scale Value
- Quantization error = $\frac{V_R}{2^n}\%$
- Flash type ADC: $2^{n-1} \rightarrow$ comparators

  $2^n \rightarrow$ resistors

  $2^n \times n \rightarrow$ Encoder

$$T_C = \frac{2^n}{2} \times \frac{1}{f_{clock}} = 2^{n-1} \times \frac{1}{f_{clock}}$$

**Fastest ADC:**

- Successive approximation ADC: n clock pulses.
- Counter type ADC: $2^n - 1$ clock pulses
- Dual slope integrating type: $2^{n+1}$ clock pulses.

**Digital to Analog Converter (DAC):**

- $FSV = V_R\left(1 - \frac{1}{2^n}\right)$
- Resolution $= \frac{\text{Step size}}{FSV} = \frac{V_R / 2^n}{V_R\left(1 - \frac{1}{2^n}\right)} = \frac{1}{2^n - 1} \times 100\%$
- Accuracy $= \pm\frac{1}{2}LSB = \pm\frac{1}{2^{n+1}}$
- Analog output = K. digital output

## FSM & Memory

**Finite State Machines (FSM):**

A general block diagram of clocked sequential circuit is also known as finite stable machine (FSM). Depending upon the external outputs, there are two types of models of sequential circuits.

1. Mealy Model
2. Moore Model

**Mealy model:**

In mealy model, the next state of the function depends on present state as well as present inputs.

**Moore Model:**

In moore model, the next state depends on the present state and present inputs but also on the outputs of more model.

The block diagram of a Moore model is given as

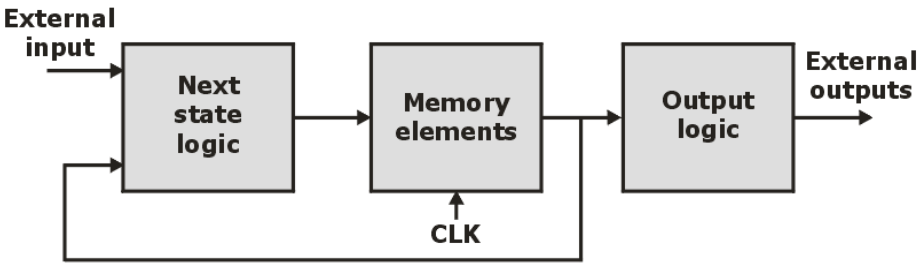



**Fig. 24**

**MEMORY:**

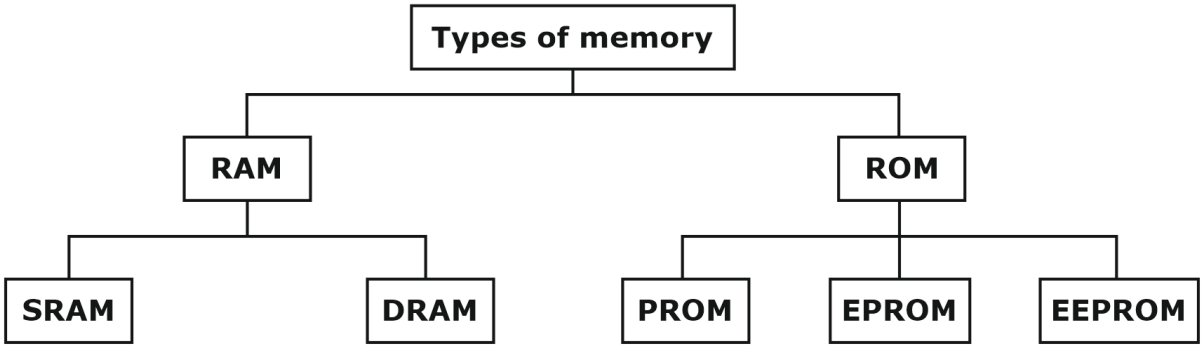



**Fig. 25**

**RAM:**

Below table lists some of the differences between SRAM and DRAM:

| SRAM | DRAM |
|---|---|
| 1. SRAM has lower access time, so it is faster compared to DRAM. | 1. DRAM has higher access time, so it is slower than SRAM. |
| 2. SRAM is costlier than DRAM, | 2. DRAM costs less compared to SRAM. |
| 3. SRAM requires constant power supply, which means this type of memory consumes more power. | 3. DRAM offers reduced power consumption, due to the fact that the information is stored in the capacitor. |
| 4. Due to complex internal circuitry, less storage capacity is available compared to the same physical size of DRAM memory chip. | 4. Due to the small internal circuitry in the one-bit memory cell of DRAM, the large storage capacity is available. |
| 5. SRAM has low packaging density. | 5. DRAM has high packaging density. |

**Table 16: Difference between SRAM and DRAM**

**ROM:**

The required paths in a ROM may be programmed in four different ways.

**Mask programming:** fabrication process

**Read-only memory or ROM:** blown fuse/fuse intact

**Erasable PROM or EPROM:** placed under a special ultraviolet light for a given period will erase the pattern in ROM.

**Electrically erasable PROM(EEPROM):** erased with an electrical signal instead of ultraviolet light.

**PROM, PAL & PLA:**

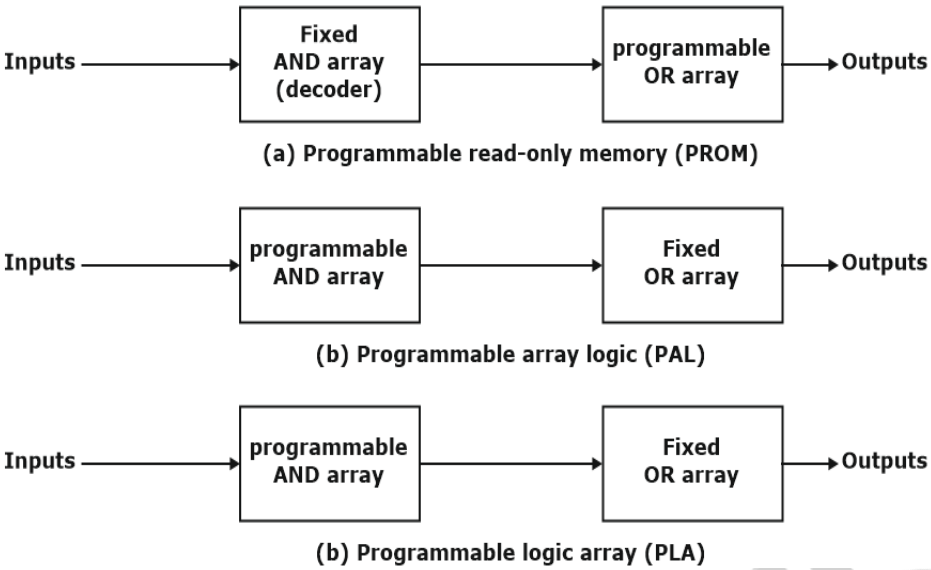



**Fig. 26**

| AND | OR | |
|---|---|---|
| Fixed | Programmable | PROM |
| Programmable | Fixed | PAL |
| Programmable | Programmable | PLA |

**Table 17: AND & OR Gates in PROM, PAL and PLA**

**Logic Family**

- Figure of Merit, $\boxed{FOM = T_{PD} \times P_D}$

  where, $T_{PD}$ is the average transition delay time for the signal to propagate from input to output of logic gate.

  and, $P_D$ is the power drawn from power supply.

- Fan out of a logic gate = $\frac{I_{OH}}{I_{IH}}$ or $\frac{I_{OL}}{I_{IL}}$

- Noise margin: $NM_H = V_{OH} - V_{IH}$ or $NM_L = V_{OL} - V_{IL}$

  $\boxed{V_{OH} > V_{IH} > V_{IL} > V_{OL}}$

  $\boxed{\text{Noise-Margin} = \min(NM_H, NM_L)}$

- Logic swing: $V_{OH}$ - $V_{OL}$
- Power Dissipation,

  $P_D = V_{CC}\, I_{CC} = V_{cc}\left[\frac{1 +_ I}{2}\right]$

  $I \to I_c$ when output low.
  $I \to I_c$ when output high.

- TTL , ECL & CMOS are used for MSI or SSI.

- $I^2L$ has best FOM.
- RTL , DTL , TTL → saturated logic
  ECL → Unsaturated logic
- Advantages of Active pullup; increased speed of operation, less power consumption.
- For TTL floating input is considered as logic "1" & for ECL it is logic "0".
- "MOS" mainly used for LSI & VLSI. Fan out is too high.
- ECL is the fastest gate & consumes more power.
- CMOS is the slowest gate & less power consumption.
- NMOS is faster than CMOS.
- Gates with open collector output can be used for wired AND operation (TTL).
- Gates with open emitter output can be used for wired OR operation (ECL).
- 1024 × 8 memory can be obtained by using 1024 × 2 memories.
- Number of memory ICs of capacity 1k × 4 required to construct memory of capacity 8k × 8 are "16"

*****

BYJU'S EXAM PREP

# Electrical Engineering

## Power System

SHORT NOTES

# POWER SYSTEM (FORMULA NOTES)

## 1. PER UNIT SYSTEM

$$\text{Quantity in per unit} = \frac{\text{actual quantity}}{\text{base value of quantity}}$$

$$\text{Quantity in percent} = \frac{\text{actual quantity}}{\text{base value of quantity}} \times 100$$

**Advantages of pu system:**

- Network analysis is made simple since all impedances of a given equivalent circuit can directly be added together regardless of the system voltages.
- It eliminates the √3 multiplications and divisions that are required while dealing with balanced three-phase systems.
- Differences in operating characteristics of many electrical apparatus can easily be estimated by comparison of their per unit values.

(i) Base value of Impedance $Z_B = \frac{(kV_B)^2}{(MVA)_B}$

Where, $kV_B$ = voltage base in kilovolts,

$(MVA)_B$ = volt-ampere base in MVA

$$Z_{pu} = \frac{\left(Z_{\text{Actual value}}\right)}{\left(kV_B\right)^2} MVA_B$$

Similarly,

$$I_{p.u.} = \frac{I_{\text{actual value}}}{I_{\text{Base}}} \quad \text{or} \quad V_{p.u.} = \frac{V_{\text{Actual value}}}{V_{\text{Base}}}$$

Change of base: $Z_{pu(new)} = Z_{pu(old)} \left[\frac{MVA_{B(new)}}{MVA_{B(old)}}\right] \left[\frac{kV_{B(old)}}{kV_{B(new)}}\right]^2$

## 2. POWER TRANSMISSION

Although copper has a much higher conductivity, aluminum conductors are normally used for overhead transmission lines. An aluminum conductor is lighter, cheaper, and has a larger diameter than a copper conductor of the same resistance.

**Conductor types**:

| S.No | Property | Solid conductor | Stranded Conductor (AAC) | Composite Stranded conductor (ACSR) |
|---|---|---|---|---|
| 1 | Skin effect | very high | Very less | Moderate |
| 2 | Mechanical strength | Very high | Very less | moderate |

**Note Points:**

- Bundled conductors is used to reduce corona loss
- Number of small conductors in composite structure $N=3n^2+3n+1$, where n is no. of layers
- Effective diameter of composite conductor D=(2n+1)d, d is diameter of small conductor.
- Skin effect depends on
    i. Frequency
    ii. Permeability
    iii. Conductivity
    iv. Radius of the conductor
- Skin depth has inverse relation with skin effect

**Inductance of the Transmission line:**

**(i)Inductance of a Conductor due to Internal Flux:**

A current carrying conductor whose cross section is of a long cylindrical structure with radius 'r' metres and carrying a current 'I' Amperes then

$$\psi_{int} = \int_0^r \frac{\mu I x^3}{2\pi r^4} = \frac{\mu I}{8\pi} \quad \text{WbT/m}$$

$$L_{int} = \int_0^r \frac{\mu I x^3}{2\pi r^4} = \frac{\mu}{8\pi} \text{ H/m}$$

If relative permeability $\mu_r = 1$, then permeability $\mu$ which is equal to $\mu_0\mu_r$ is given by

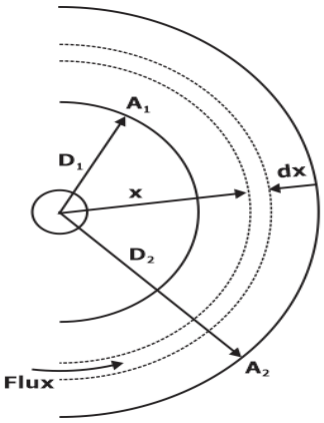
$\mu = \mu_0\mu_r = 4\pi \times 10^{-7}$ H/m

Hence, $\psi_{int} = \frac{I}{2} \times 10^{-7}$ WbT/m

$L_{int} = \frac{1}{2} \times 10^{-7}$ H/m

**(ii)Inductance of a Conductor due to External Flux:**

A same conductor with cross section of a conductor be as radius 'r' metres and carrying a current 'I' Amperes

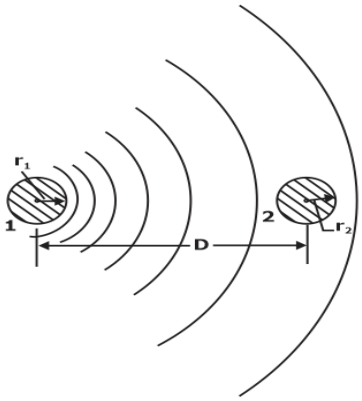



The inductance due to the external flux linkages included between $A_1$ and $A_2$ is

$$L_{12} = 2 \times 10^{-7} \ln \frac{D_2}{D_1} \text{ H/m}$$

**(iii)Inductance of a single-phase, two-wire system**

Consider a single-phase circuit of two parallel conductors of radii $r_1$ and $r_2$ metres, separated by a distance 'D' metres, as shown in figure below, one conductor is the return circuit for the other.



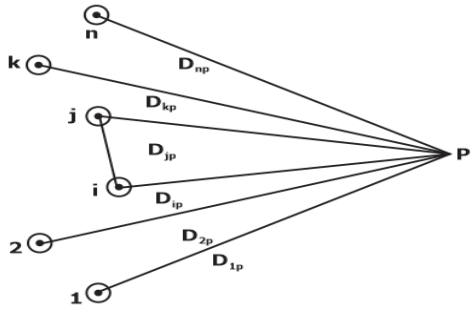

The total inductance of the circuit taking $r_1' = r_2' = r'$ can be written as,

$$L = L_1 + L_2 = 4 \times 10^{-7} \ln \frac{D}{r'} \text{ H/m}$$

where $r_1' = 0.7788 r_1$ is the geometric mean radius (GMR) or the self-geometric mean distance (self-GMD) of a solid round conductor.

**(iv) Flux linkages of one conductor in a group of conductors**

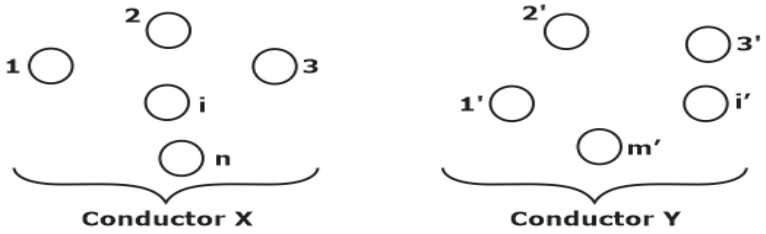
Consider a group of 'n' conductors, as shown in figure below, carrying phasor currents $I_1$, $I_2$, ..., $I_n$, whose sum equals zero.




$$\psi_i = 2\times10^{-7}\left(I_1 \ln\frac{1}{D_{i1}} + I_2 \ln\frac{1}{D_{i2}} + \ldots + I_i \ln\frac{1}{r'_i} + \ldots + I_n \ln\frac{1}{D_{in}}\right)$$

**(v) Inductance of composite conductor lines**

Consider a single-phase line composed of conductor X, having 'n' identical parallel filaments, and conductor Y, which is the return circuit for the current in conductor X, having 'm' identical parallel filaments, as shown in figure below.




The inductance of conductor X, therefore, given as

$$L_X = 2\times10^{-7}\ln\frac{\left[\begin{matrix}(D_{11'}D_{12'}D_{13'}\ldots D_{1m'})(D_{21'}D_{22'}D_{23'}\ldots D_{2m'})\ldots\\(D_{i1'}D_{i2'}D_{i3'}\ldots Dim')\ldots(D_{n1'}D_{n2'}D_{n3'}\ldots D_{nm'})\end{matrix}\right]^{1/mn}}{\left[\begin{matrix}(D_{11}D_{12}D_{13}\ldots D_{1n})(D_{21}D_{22}D_{23}\ldots D_{2n})\ldots\\(D_{i1}D_{i2}D_{i3}\ldots D_{in})\ldots(D_{n1}D_{n2}D_{n3}\ldots D_{nn})\end{matrix}\right]^{1/n^2}} \text{ H/m}$$

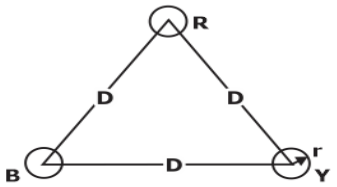
**Bundled Conductors:**

For EHV and UHV lines, bundled conductors having two or more conductors per phase are used. The use of bundled conductors reduces corona loss and decreases the reactance of the transmission line. The reduction of reactance results in increased GMR of the bundle.

**Inductance of Three Phase lines:**

**(i) Symmetrical Spacing:**

Consider an arrangement of equilaterally spaced conductors in a three-phase circuit which is shown in figure below,

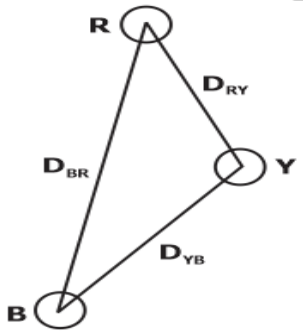



Each conductor has a radius 'r' metre and the spacing between the conductors is 'D' metres. Assume that the neutral wire is not present, hence, $I_R + I_Y + I_B = 0$.

$L_R = \frac{\psi_R}{I_R} = 2 \times 10^{-7} \ln \frac{D}{r'}$ H/m . The values of inductances for conductor Y & B are the same because of symmetry.

**(ii) Unsymmetrical Spacing:**

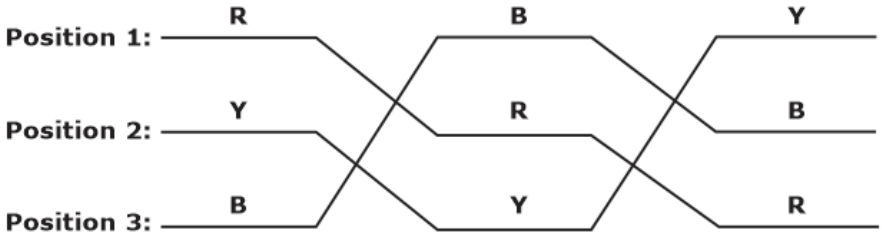
Consider an arrangement of conductors in a three-phase circuit (with unsymmetrical spacing) as shown in figure below,




Unsymmetrical spacing of the phase conductors of a three-phase line causes the flux linkages and inductance of each phase to be different and results in an unbalanced circuit.

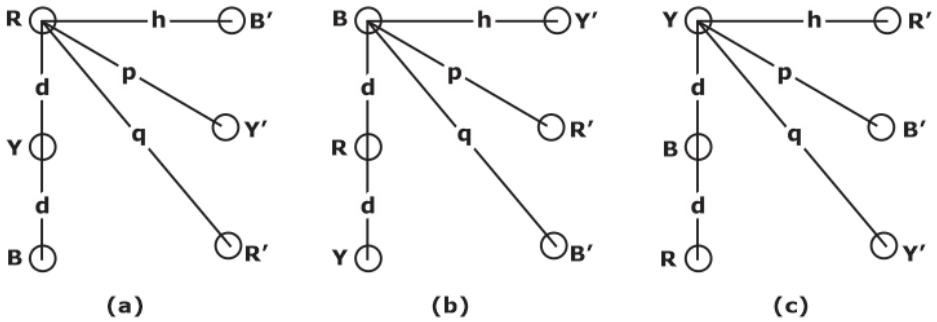
**Transposition:** The three phases can be balanced by exchanging the positions of the conductors at regular intervals along the line so that each conductor occupies the original position of every other conductor over an equal distance. Such an exchange of conductor positions is called transposition.

For a complete transmission cycle, the transposed lines are shown in figure below, where conductors are designated as R, Y, and B, and the positions occupied are numbered 1, 2, and 3, respectively.




**Inductance of double circuit lines:**

Consider an arrangement of conductors in a double circuit three phase transposed line as shown in figure below, keeping each phase at a different place i.e. Phase R in position 1 (a), Phase R in position 2 (b), and Phase R in position 3 (c).




The conductors R, Y, B belong to one circuit while conductors R′, Y′, B′ belong to another circuit. Inductance of each phase is given by

$$L = 2\times10^{-7}\ln\left[2^{1/6}\left(\frac{D}{r'}\right)^{1/6}\left(\frac{p}{q}\right)^{1/3}\right]\ \text{H/m}$$

**Capacitance of the transmission line**

**(i) Capacitance of a Two-conductor Line:**

Consider an arrangement of a two-conductor line connected to a single-phase ac supply as shown in figure below. The radii of the conductors are $r_a$ and $r_b$, and the charges on the conductors are $q_a$ and $q_b$, respectively. The distance between the two conductors is D.

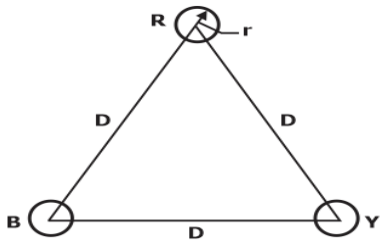
**Assumption:** The ground is far away from the conductors and the charge distribution over the surface of the conductors is uniform.

The capacitance $C_{ab}$ between the conductors is given by

$$C_{ab} = \frac{q_a}{V_{ab}} = \frac{2\pi\varepsilon}{\ln\left(\frac{D^2}{r_a r_b}\right)} = \frac{\pi\varepsilon}{\ln\left(\frac{D}{\sqrt{r_a r_b}}\right)}\ \text{F/m}$$

**(ii) Capacitance of a Three-phase Line with Equilateral Spacing:**

Consider a three-phase line with the conductors equilaterally spaced as shown in figure below. The three-phase conductors R, Y, and B have equal cross sections. Let the distance between the conductors be D, and the charges on the conductors be $q_R$, $q_Y$, and $q_B$ respectively.




The capacitance to neutral $C_n$, therefore, is

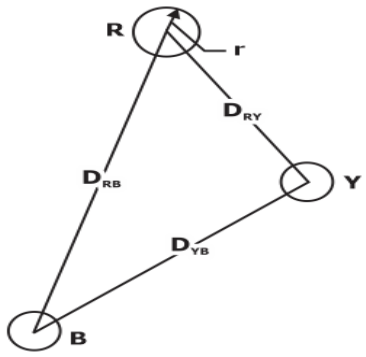
$$C_n = \frac{q_R}{V_{RN}} = \frac{2\pi\varepsilon}{\ln\frac{D}{r}} \text{ F/m}$$

The charging current per phase is

$$I_{ch} = j\omega C_n V_{RN} = j\omega \times \frac{2\pi\varepsilon}{\ln\frac{d}{r}} \times V_{RN} \quad \text{A/m}$$

**(iii)Capacitance of a Three-phase Line with Unsymmetrical:**

Consider an arrangement of a three-phase, fully transposed line with three identical conductors of radius 'r' with unequal spacing as shown in figure below,




Hence, capacitance to neutral $C_n$

$$C_n = \frac{q_R}{V_{RN}} = \frac{2\pi\varepsilon}{\ln\frac{D_{eq}}{r}} \text{ F/m}$$

$$= \frac{0.02412}{\ln\frac{D_{eq}}{r}} \text{ μF/km}$$

Also, charging current is given as

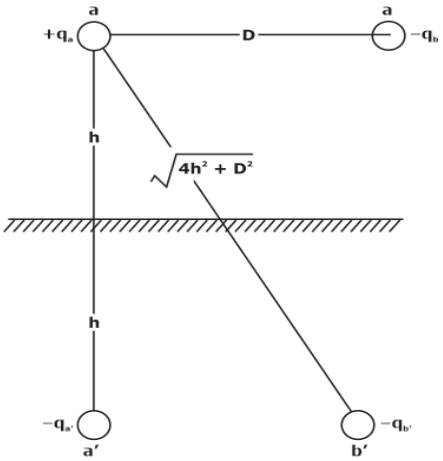
$$I_{ch} = j\omega C_n V_{RN} = j\omega \left( \frac{2\pi\varepsilon}{\ln\frac{D_{eq}}{r}} \right) \times V_{RN} \quad \text{A/m}$$

Where,

$$D_{eq} = \sqrt[3]{D_{RY} D_{YB} D_{BR}}$$

**(iv)Capacitance of a Single-phase Line with effect of earth:**

Consider a single-phase charged line in the presence of the ground as shown in figure below.




Capacitance of the single-phase line, including the effect of earth, is given by

$$C_{ab} = \frac{\pi\varepsilon}{\ln \frac{D}{r\sqrt{\left\{1+\frac{D^2}{4h^2}\right\}}}} \text{ F/m}$$

Line-to-neutral capacitance of the line, is given as

$$C_n = \frac{2\pi\varepsilon}{\ln \frac{D}{r\sqrt{\left\{1+\frac{D^2}{4h^2}\right\}}}} \text{ F/m}$$

## 3. PERFORMANCE OF LINES

Performance of lines is meant for the determination of efficiency and regulation of lines.

(i) The efficiency of lines is defined as

$$\% \text{ efficiency} = \frac{\text{Power delivered at the receiving end}}{\text{Power sent from sending end}} \times 100$$

$$\% \text{ efficiency} = \frac{\text{Power delivered at the receiving end}}{\text{Power delivered at the receiving end} + \text{losses}} \times 100$$

(ii) Regulation of lines is defined as a ratio of difference between no load and full load voltage to full load voltage.

$$\text{Regulation up} = \frac{V_r' - V_r}{V_r}$$

Regulation down $= \frac{V_r' - V_r}{V_r'}$, we use Regulation UP in Transmission lines

Where $V_r'$ is the receiving end voltage under no-load condition and $V_r$ is the Receiving end voltage under full load condition.

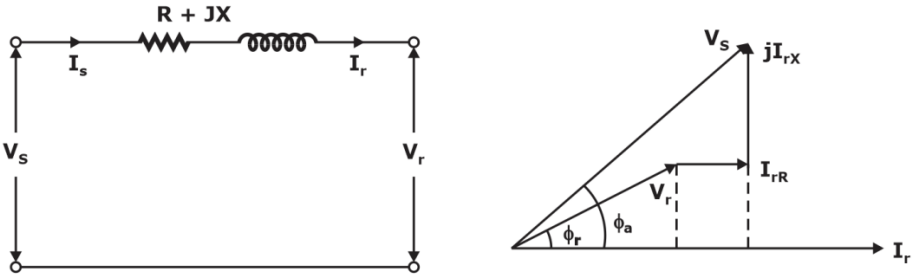
**Parameters Table**:

| S. No. | Line Description | R | L | $X_L$ | C | $X_C$ |
|---|---|---|---|---|---|---|
| 1. | Length Increases | Increases | Increases | Increases | Increases | Decreases |
| 2. | Distance of separation | No change | Increases | Increases | Decreases | Increases |
| 3. | Radius of conductor | Decreases | Decreases | Decreases | Increases | Decreases |
| 4. | Symmetrical spacing | Does not depend | Decreases | Decreases | Increases | Decreases |
| 5. | Unsymmetrical spacing | Does not depend | Increases | Increases | Decreases | Increases |
| 6. | Effect of earth is taken into account | No change | No change | No change | Increases | Decreases |
| 7. | Height of the conductor increases | No change | No change | No change | Decreases | Increases |

## Short Transmission Line:

- Based on frequency and length short line defined as: If $< 4000hz - Km$
- Based on Voltage short line defined as: V=0-20kV

The equivalent circuit and vector diagram for a short transmission line are shown in fig.




By the phasor diagram

$|V_s|\cos(\phi_a - \phi_r) = |V_r| + I_r Z\cos(\theta - \phi_r)$

Where,

$\phi_a$ is sending end power factor

$\phi_r$ is receiving end power factor and $\theta$ is impedance angle.

For short Transmission line, $\phi_a$ - $\phi_r$ is low so we can approximate equation like

$$|V_s| \simeq |V_r| + I_r Z\cos(\theta - \phi_r)$$

The receiving end voltage under no load is the same as the sending end voltage under full load condition.

$$\% \text{ regulation} = \frac{V_s - V_r}{V_r} \times 100 = \left(\frac{I_r R}{V_r}\cos\phi_r + \frac{I_r X}{V_r}\sin\phi_r\right) \times 100$$

$$\text{Regulation per unit} = \frac{I_r R}{V_r}\cos\phi_r \pm \frac{I_r X}{V_r}\sin\phi_r = R\cos\phi_r \pm X\sin\phi_r$$

+ for lagging and – for leading

- Regulation is positive for lagging power factor and positive, negative or zero for leading power factors.
- Maximum regulation occurs at $\theta = \phi_r$
- Zero regulation occurs when $\theta + \phi_r = \frac{\pi}{2}$

` ABCD Parameters equations:

$V_s = AV_r + BI_r$

$I_s = CV_r + DI_r$

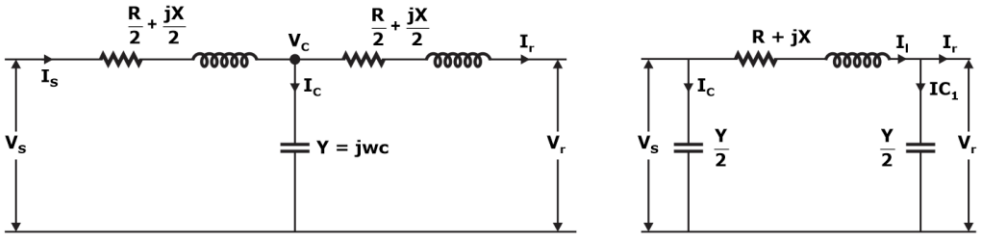
- The constants for short transmission lines are,

  A = 1, B = Z, C = 0, D = 1

- $\% \text{ regulation} = \frac{\left|\frac{V_s}{A}\right| - V_r}{V_r} \times 100$
- $\% \eta = \frac{\text{Power received at the receiving end}}{\text{Power received per at the receiving end} + \text{losses}} \times 100$

**Medium Transmission Line:**

- Based on frequency and length short line defined as: 4000<lf<12000hz-Km
- Based on Voltage short line defined as: V=20-100kV
- The two configurations are known as nominal –T and nominal $-\pi$ respectively.




A, B, C, D constant for nominal – T

$$A = 1 + \frac{YZ}{2}$$

$$B = Z\left(1 + \frac{YZ}{4}\right)$$

$$C = Y$$

$$D = \left(1 + \frac{YZ}{2}\right)$$

A, B, C, D constants for nominal $-\pi$

$$A = 1 + \frac{YZ}{2}$$

$$B = Z$$

$$C = Y\left(1 + \frac{YZ}{4}\right)$$

$$D = \left(1 + \frac{YZ}{2}\right)$$

**Long Transmission Line:**

- Based on frequency and length short line defined as: If $> 12000hz - Km$
- Based on Voltage short line defined as: V>100KV
- ABCD equations for long transmission line

$$V_s = V_r \cosh\gamma l + I_r Z_C \sinh\gamma l$$

$$I_S = V_r \frac{\sinh\gamma l}{Z_C} + I_r \cosh\gamma l$$

- The propagation constant $\gamma = \alpha + i\beta$; the real part is known as attenuation constant and the quadrature component β the phase constant and is measured in radians per unit length.
- For loss less line $\gamma = i\beta$
- $\gamma = \sqrt{ZY}$ and $\beta = \omega\sqrt{LC}$
- Wavelength $\lambda = \frac{2\pi}{\beta}$
- Since the equivalent circuit model of both the short- and medium length lines are passive, linear, bilateral, and time invariant, AD - BC = 1
- Approximate ABCD of long Transmission Line:

$$A=D=\left(1+\frac{ZY}{2}\right)$$

$$B=Z\left(1+\frac{ZY}{6}\right)$$

$$C=Y\left(1+\frac{ZY}{6}\right)$$

- A, B, C, D constant for Equivalent – T

$$A=1+\frac{Y'Z'}{2}$$

$$B=Z'\left(1+\frac{Z'Y'}{4}\right)$$

$$C=Y'$$

$$D=\left(1+\frac{Y'Z'}{2}\right)$$

Where, $Y'=\frac{1}{Z_c}\sinh\gamma l=\frac{i}{Z_c}\sin\beta l$ and $\frac{Z'}{2}=Z_c\left(\frac{\cosh\gamma l-1}{\sinh\gamma l}\right)=-iZ_c\left(\frac{\cos\beta l-1}{\sin\beta l}\right)$

- A, B, C, D constants for Equivalent– $\pi$

$$A=1+\frac{Y'Z'}{2}$$

$$B=Z'$$

$$C=Y'\left(1+\frac{Y'Z'}{4}\right)$$

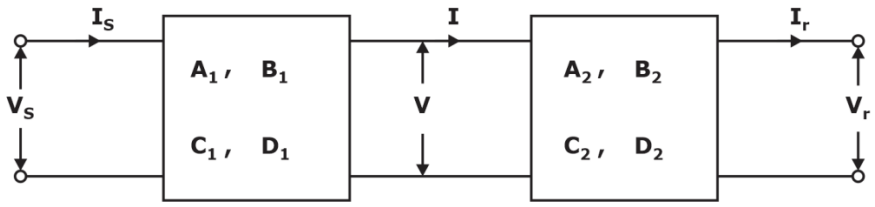
$$D=\left(1+\frac{Y'Z'}{2}\right)$$

Where,

$$\frac{Y'}{2}=\left(\frac{\cosh\gamma l-1}{Z_c\sinh\gamma l}\right)=\frac{-i}{Z_c}\frac{\cos\beta l-1}{\sin\beta l}$$

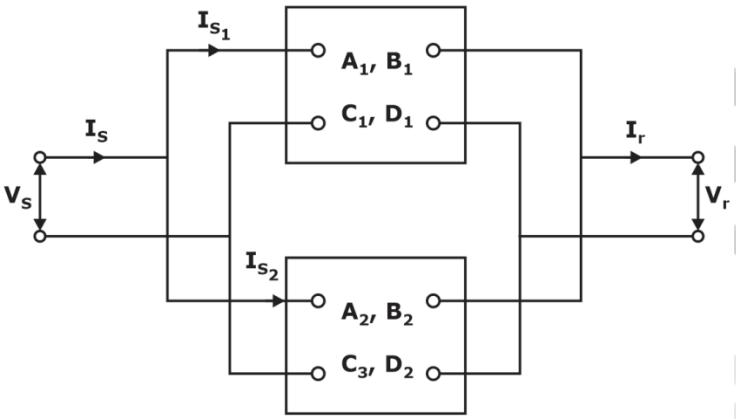
$$Z'=Z_c\sinh\gamma l=iZ_c\sin\beta l$$

**Constants for Two networks in Tandem**




$$\text{Equivalent}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

**Constants for networks in parallel**




$$\text{Equivalent Single Network Parameters} \begin{cases} A = \dfrac{A_1B_2 + A_2B_1}{B_1 + B_2} \\ B = \dfrac{B_1.B_2}{B_1 + B_2} \\ A = D = \dfrac{A_1B_2 + A_2B_1}{B_1 + B_2} = \dfrac{D_1B_2 + D_2B_1}{B_1 + B_2} \\ C = C_1 + C_2 + \dfrac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \end{cases}$$

**Surge impedance loading of lines:**

- Surge impedance loading (SIL) is the power delivered at rated voltage by a lossless line to a load resistance whose value is equal to the surge impedance $Z_c = \sqrt{Z_{oc}Z_{sc}} = \sqrt{\frac{L}{C}}$.
- For a lossless line, $Z_C$ is purely resistive.
- At the rated line voltage, the surge impedance loading of the line is given by

$$SIL = \frac{V_{R(rated)}^2}{Z_c}$$

where rated voltage is used for a single-phase line and rated line-to-line voltage is used for a three-phase line.

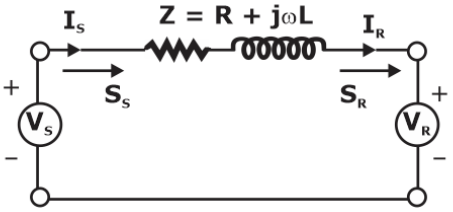
- The surge impedance loading of the line in itself is not a measure of the maximum power that can be delivered over the line.

- To utilize the transmission line effectively, loading is greater than SIL will be economical on overhead line but for stability reasons always loading is less than maximum power transfer capability ($P_{max}$).
- Loading on the underground cable done based on thermal stability. So loading is less than SIL is economical in Underground cable

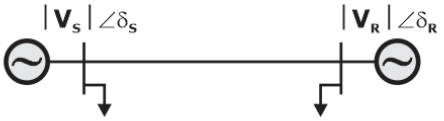
| Loading= SIL | Loading > SIL | Loading < SIL |
|---|---|---|
| $Z_L=Z_C$ | $Z_L<Z_C$ | $Z_L>Z_C$ |
| $\|V_r\| = \|V_s\|$ | $\|V_r\| < \|V_s\|$ | $\|V_r\| > \|V_s\|$ |
| Neither sink nor source of Q | Transmission line is sink of Q | Transmission line is source of Q |

**Complex power transmission in short-length line**

Figure shown below represents a per-phase circuit of a short transmission line.




**(a) Per-phase circuit diagram for a short-length line**




**(b) Equivalent circuit for a short-length line**

$$P_s = \frac{|V_s|^2}{|Z|}\cos\theta - \frac{|V_s||V_R|}{|Z|}\cos(\theta+\delta)$$

$$Q_s = \frac{|V_s|^2}{|Z|}\sin\theta - \frac{|V_s||V_R|}{|Z|}\sin(\theta+\delta)$$

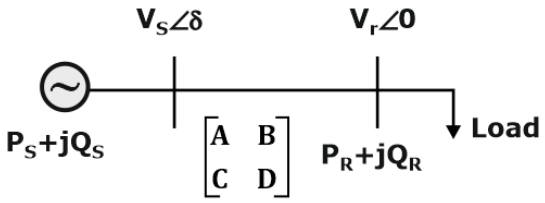
where $P_s$ and $Q_s$ are the active and reactive power flows, respectively, at the sending end and $\theta$ is impedance angle

$$P_R = -\frac{|V_R|^2}{|Z|}\cos\theta + \frac{|V_R||V_S|}{|Z|}\cos(\theta-\delta)$$

$$Q_R = -\frac{|V_R|^2}{|Z|}\sin\theta + \frac{|V_R||V_S|}{|Z|}\sin(\theta-\delta)$$

$P_R$ and $Q_R$ are the active and reactive power flows, respectively, at the receiving end

**Power Equations in ABCD Parameters**




Real and Reactive powers at receiving end is $P_R$ and $Q_R$

$$P_R = \frac{|V_S||V_R|}{|B|}\cos(\beta-\delta) - \frac{|A|}{|B|}|V_R|^2\cos(\beta-\alpha)$$

$$Q_R = \frac{|V_S||V_R|}{|B|}\sin(\beta-\delta) - \frac{|A|}{|B|}|V_R|^2\sin(\beta-\alpha)$$

$$\left.\begin{aligned} A &= |A|\angle\alpha \\ B &= |B|\angle\beta \\ C &= |C|\angle\gamma \\ D &= |D|\angle\Delta \end{aligned}\right\} \text{(ABCD Parameters)}$$

**Condition for maximum power transfer**

- Maximum at $\beta = \delta$

$$P_{max} = \frac{|V_S||V_R|}{|B|} - \frac{|A|}{|B|}|V_R|^2\cos(\beta-\alpha)$$

$$Q_{R,max} = -\frac{|A|}{|B|}|V_R|^2\sin(\beta-\alpha)$$

- Always active power flows from leading voltage bus to lagging voltage bus provided that system is loss less
- Always reactive power flows from higher magnitude voltage terminal to lower magnitude voltage terminal provided that both voltages are taken on same reference.

## 4. VOLTAGE CONTROL DEVICES AND APPLICATIONS

| **Static voltage control Device** | **Application** |
|---|---|
| Shunt Capacitor | i)Power factor correction Device<br>ii)Used at the places where under voltage occurs |
| Shunt Reactor | i)Practically Ferranti effect controller<br>ii)Used to reduce voltage magnitude at the buses where overvoltage occurs under steady state operation<br>iii)Improves power factor if load is leading type, but practically most loads are lagging type in system |
| Series Capacitance | i)used to rise in voltage magnitude when load is lagging type and reduce in voltage magnitude when load is leading type<br>ii)steady state stability improved |
| Series Reactor | Practically used as fault current limiter as smoothing reactor in HVDC & PE circuits |
| **Dynamic voltage control Devices** | |
| Synchronous condenser | Over excited synchronous motor under no load condition. It is similar to shunt capacitor |
| Synchronous Coil | Under excited synchronous motor under no load condition. It is similar to Shunt Reactor |
| Synchronous Phase modifier | It is a synchronous motor that runs without mechanical load. It can be used to vary power factor by altering excitation |

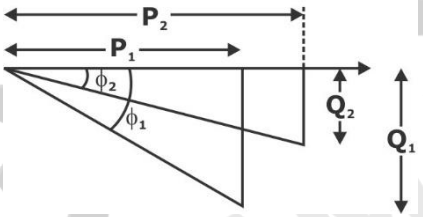
- Rating of Shunt capacitor bank to boost up voltage $Q_{sh,cap} = \frac{(\Delta V_c)|V_s|}{X_{thevnin}}$
- Rating of Shunt reactor bank to reduce the voltage $Q_{sh,reactor} = \frac{(\Delta V_L)|V_s|}{X_{thevnin}}$
- For same voltage magnitude rise rating of shunt capacitor is more than the series capacitor
- Degree of compensation: The characteristic impedance of uncompensated line is given by

$$Z_c = \sqrt{\frac{X_{line}}{B_{line}}}$$

If we add a series capacitive reactance with line and shunt inductive reactance at load then the characteristic impedance of compensated line is given by

$$Z_c = \sqrt{\frac{X_{line} - X_c}{B_{line} - B_L}}$$

- By synchronous phase modifier, the power factor got improved due to reactive power injected by the machine.




$$Q_M = Q_1 - Q_2$$

$$= S_1 \sin\phi_1 - S_2 \sin\phi_2$$

$$Q_M = P_1 \tan\phi_1 - P_2 \tan\phi_2$$

## 5.REFLECTION AND REFRACTION OF WAVES

- Forward current and voltage are in same phase and backward current and voltage are in opposite phase

  Forward voltage = $Z_c$ (Forward current)

  Backward voltage = $-Z_c$ (Backward current)
- Incident + Reflection = Refraction
- $Z_L$ and $Z_C$ are load and characteristic impedance then

  Coefficient of Voltage refraction is $V_{refraction} = \frac{2Z_L}{Z_L + Z_C}$

  Coefficient of Voltage reflection is $V_{reflection} = \frac{Z_L - Z_C}{Z_L + Z_C}$

Coefficient of Current refraction $I_{refraction} = \frac{2Z_C}{Z_L + Z_C}$

Coefficient of Current reflection $I_{reflection} = -\left(\frac{Z_L - Z_C}{Z_L + Z_C}\right)$

- If the load is inductor, then

Refracted voltage equation is $V(t) = 2Ve^{\left(\frac{-Z_c t}{L}\right)}$

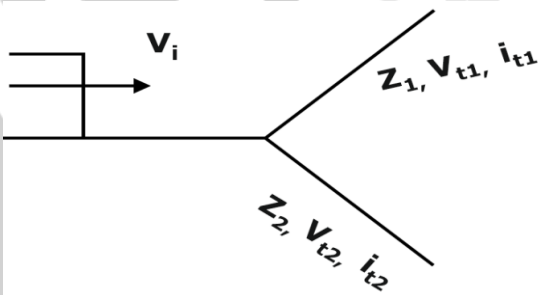
Refracted current equation is $i(t) = -\frac{2V}{Z_c} e^{\left(\frac{-Z_c t}{L}\right)}$

- If the load is capacitor, then

Refracted voltage equation is $V(t) = 2V\left(1 - e^{\left(\frac{-t}{CZ_c}\right)}\right)$

Refracted current equation $i(t) = \frac{2V}{R} e^{\left(\frac{-t}{CZ_C}\right)}$

**Forked Line:**




Line terminated with two different impedances ($Z_1$ and $Z_2$) which are in parallel. So combined $Z_L$ is parallel combination of $Z_1$ and $Z_2$.

In above formulas $Z_L$ is replaced by new $Z_L$ i.e., $Z_L = \frac{Z_1 Z_2}{Z_1 + Z_2}$

- Highest switching voltage experienced by the line for ac supply is $2V_{Phase}$ (Peak value)
- In general switching overvoltage in the range of 2.5p.u -3.5p. u
- If surge propagates from Overhead (high impedance) line to cable (low impedance) then
    1. Surge voltage magnitude decreases
    2. Surge current Increases

3. Rate of rise of surge voltage (Steepness) decreases
4. Steepness in surge current increases

- If surge propagates from cable (low impedance) line to overhead (high impedance) then
  1. Surge voltage magnitude increases
  2. Surge current decreases
  3. Rate of rise of surge voltage (Steepness) increases
  4. Steepness in surge current decreases
- Velocity of wave propagation in Overhead line= 3 x $10^8$ m/sec
- Velocity of wave propagation in cable

$$v = \frac{3\times10^8}{\sqrt{\varepsilon_r}} \text{ m/sec as } \varepsilon_r > 1$$ Velocity of wave in cable is less than overhead line

| Coefficient | Open circuit line ($Z_L=\infty$) | Short circuit line ($Z_L=0$) | Flat line ($Z_L=Z_C$) |
|---|---|---|---|
| V refraction | 2 | 0 | 1 |
| V reflection | 1 | -1 | 0 |
| I refraction | 0 | 2 | 1 |
| I reflection | -1 | 1 | 0 |

## 6. INSULATORS

**Pin type insulators:** Pin type insulators are used for transmission and distribution of electric power at voltages up to 33 kV.

**Suspension type insulators**: Generally, it is made up of porcelain material that includes single or a string of insulating discs hanged over a tower. It operates at above 33KV and overcomes the limitation of pin type insulator

**Strain insulators**: When there is a dead end of the line or there is sharp curve, the line is subjected to greater tension. Therefore, strain insulators are used

**Shackle insulators**: Shackle type insulators used for low voltage distribution lines and can be used either in a horizontal position or in a vertical position.

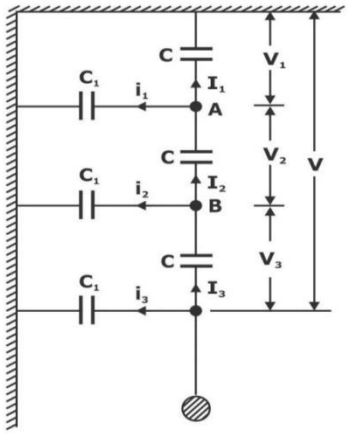
**String Efficiency:**

The ratio of voltage across the whole string to the product of number of discs and the voltage across the disc nearest to the conductor is known as string efficiency,

$$\text{String efficiency} = \frac{\text{Voltage across the string}}{n \times \text{Voltage across disc nearest to conductor}}$$

Where n = number of discs in the string.

An equivalent circuit for a 3-disc string, with self-capacitance of each disc is 'C' and shunt capacitance $C_1$ is some fraction K of self-capacitance i.e., $C_1 = KC$.




Voltage across top unit,

$$V_1 = \frac{V}{(1+K)(3+K)}$$

Voltage across second unit from top, $V_2 = V_1 (1 + K)$

Voltage across third unit from top, $V_3 = V_1 (1 + 3K + K^2)$

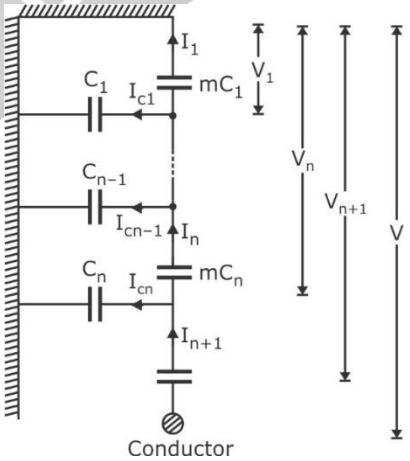
$$\%\,\text{string efficiency} = \frac{\text{Voltage across string}}{n \times \text{Voltage across disc nearest to conductor}} \times 100 = \frac{V}{3 \times V_3} \times 100$$

**Methods of Improving String Efficiency:**

The various methods for improving string efficiency are as follows:

**1.** By using longer cross-arms

**2.** By grading the insulators




If discs of different capacitances are used such that the product of their capacitive reactance and voltage across the disc is same so that the current flowing through the respective unit is same.

At junction n

$I_{n+1} = I_n + I_{cn}$

$v\omega C_{n+1} = v\omega C_n + nv\omega C$

$C_{n+1} = C_n + nC$

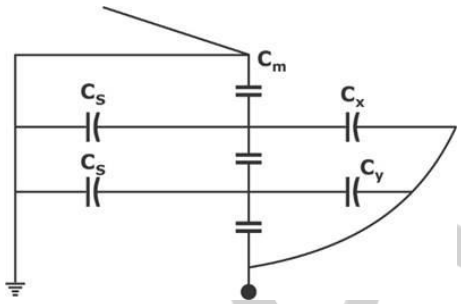
It shows as we move towards power conductor, capacitance increases.

To carry out this method, units of different capacities are required, which is uneconomical and impractical.

Hence, this method is used only for high voltage lines.

**3.** By using a guard ring or static shielding:

In this, the leakage current between the pin to the tower is cancelled out by the method of guard ring




In a n disc insulator,

Capacitance of $p^{th}$ metal link to guard wire $C_p = \left(\frac{p}{n-p}\right)C_s$ or $C_p = \left(\frac{p}{n-p}\right)kC_m$

Where $C_s$=shunt capacitance

## 7.CORONA

The power loss due to corona is given by:

$$P = 242.2\left(\frac{f+25}{\delta}\right)\sqrt{\frac{r}{d}}\left(V - V_c\right)^2 \times 10^{-5} \text{ kW/km/phase}$$

Where f = supply frequency in Hz

V = phase-neutral voltage (r.m.s.)

$V_C$ = critical disruptive voltage (r.m.s.) per phase

Methods of reducing corona loss:

(i) By increasing conductor size

(ii) By increasing conductor spacing

**Critical disruptive voltage:**

It is the minimum voltage required between the conductors to start the ionization of air.

Conductors of radii 'r' cm and spaced 'd' cm apart. If 'V' is the phase-neutral potential, then potential gradient at the conductor surface is given by

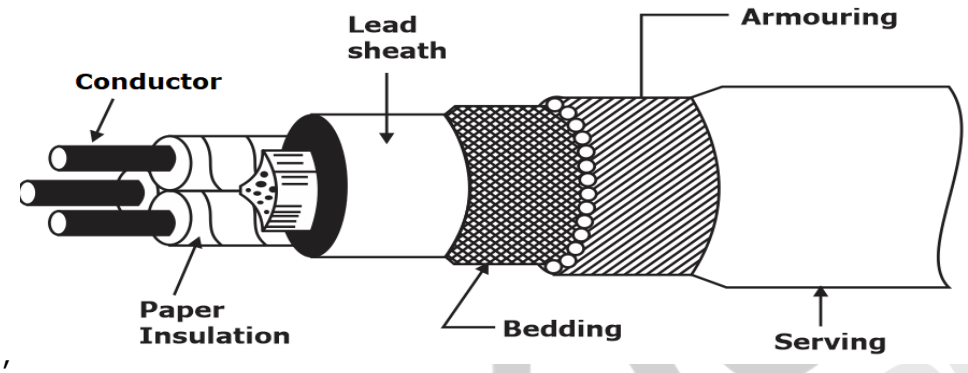
$$g = \frac{V}{r \log_e \frac{d}{r}} \text{ volts/cm}$$

In order that corona is formed, the value of 'g' must be made equal to the breakdown strength of air. The breakdown strength of air at 76 cm pressure and temperature of 25°C is 30 kV/cm (max) or 21.2 kV/cm (rms.)

## 8.UNDERGROUND CABLES

**Construction of Cables:**

Figure below shows the general construction of a 3-conductor cable:




The various parts of the cable are as follows:

**Insulation:** Each core or conductor is provided with a suitable thickness of insulation, the thickness of layer depends upon the voltage which cable needs to withstand. The commonly used materials for insulation are impregnated paper, varnished cambric, or rubber mineral compound.

**Metallic sheath:** In order to protect the cable from moisture, gases, or any other damaging liquids (such as acids or alkalis) in the soil and atmosphere, a metallic sheath of lead or aluminium is provided over the insulation.

**Bedding**: It is a paper tape compound with fibrous material (like jute, cotton, hessian cloth etc). The purpose of bedding is to protect the metallic sheath against corrosion and mechanical injury.

**Armouring:** Its purpose is to protect the cable from mechanical injury while laying it and during the course of handling.

**Serving:** In order to protect armouring from atmospheric conditions, a layer of fibrous material (like jute, PVC) similar to bedding is provided over the armouring this is known as serving.

**Properties:**

1. High insulation resistance to avoid leakage current.
2. High dielectric strength to avoid electrical breakdown of the cable.
3. High mechanical strength to withstand the mechanical handling of cables.
4. Non-hygroscopic i.e., it should not absorb moisture from air or soil.
5. Non-inflammable.

**Insulation resistance:** Insulation resistance of a cable is inversely proportional to its length. In other words, if the cable length increases, its insulation resistance decreases and vice-versa.

$$R_{ins} = \frac{\rho}{2\pi l} \ln\left(\frac{R}{r}\right) \text{ohm}$$

R= internal radius of the sheath

r= radius of the conductor

- Most economical conductor size in a cable: $r = \frac{R}{2.718}$

  And the value of $g_{max}$ under this condition is $g_{max} = \frac{2V}{d}$ volts/m

  Where R internal sheath radius and r is core radius
- Capacitance of 3-core cables:

  Per phase capacitance is given by $C_N = C_S + 3C_C$

  $C_S$ is conductor-sheath capacitances and $C_C$ is the capacitance between conductors

  Charging current**:** Let $V_P$ is the phase voltage then charging current $I_{Charging} = 2\pi f V_P C_N$
- Any two cores are connected to sheath and measure capacitance between third core and sheath gives capacitance = $C_S + 2C_C$
- Bunch all core together and measure capacitance between any one core to sheath results capacitance = $3C_S$
- Connect any one core to sheath and measure capacitance between remaining two core gives capacitance= $\frac{C_S + 3C_C}{2} = \frac{C_N}{2}$
- In capacitance grading, different dielectrics are placed between core and sheath in such a way that dielectric with high permittivity is closer to core.

  If $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$. Then $\varepsilon_1 d = \varepsilon_2 d_1 = \varepsilon_3 d_2$

  Where d is diameter of core, $d_1$ is diameter of first dielectric and so on.

  Total voltage,

  $$V = \frac{g_{max}}{2}\left[d \log_e \frac{d_1}{d} + d_1 \log_e \frac{d_2}{d_1} + d_2 \log_e \frac{D}{d_2}\right]$$
- In interseath grading, homogenous dielectric is used and divided into various layer by placing metallic sheath between core and lead sheath.
- Due to the insulation resistance power loss occurs.

  $P_d = \omega C V^2 \delta$ , Where $\delta$ is the loss angle

## 9. ECONOMIC LOAD DISPATCH

- The fuel cost of generator 'i' is represented as a quadratic function of real power generation,

  $$C_i = \alpha_i + \beta_i P_i + \gamma_i P_i^2$$

- The incremental fuel-cost $\frac{dC_i}{dP_i} = 2\gamma_i P_i + \beta_i$. Let $\frac{dC_i}{dP_i} = \lambda$ and for economic load dispatch power plant should have equal incremental cost

$$\frac{dC_1}{dP_1} = \frac{dC_2}{dP_2} = \frac{dC_i}{dP_i} = \lambda \qquad i = 1, 2, \ldots, n,$$

$C_i$ is the production cost of $i^{th}$ plant, $P_i$ is the generation of $i^{th}$ plant

- Economic Dispatch including losses: The total transmission loss is a quadratic function of the generator power outputs, which is given as

$$P_L = \sum_{i=1}^{n}\sum_{j=1}^{n} P_i B_{ij} P_j$$

The coefficients $B_{ij}$ are called loss coefficients or B-coefficients.

The economic dispatching problem is used to minimize the overall generating cost $C_i$, which is the function of plant output

Incremental cost of 'i' th unit

$$\frac{dC_i}{dP_i} = \lambda\left(1 - \frac{\partial P_L}{\partial P_i}\right) \qquad i = 1, 2, \ldots, n$$

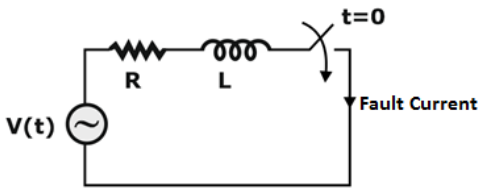
$$L_i \frac{dC_i}{dP_i} = \lambda \qquad i = 1, 2, \ldots, n$$

Where n is the total number of dispatchable generating plants. $L_i$ is known as the penalty factor of plant 'i' and is given by

$L_i = \frac{1}{1 - \frac{\partial P_L}{\partial P_i}}$; Where $\frac{\partial P_L}{\partial P_i}$ = Incremental transmission loss

## 10.FAULT ANALYSIS

Transient on Transmission line




$V(t) = V_m \sin(\omega t + \alpha)$

$$i(t) = i_{ss} + i_{transient} = \frac{V_m}{z}\sin(\omega t + \alpha - \theta) + i(0)e^{-t/\tau}$$

- At t = 0, i = 0

$$i(t) = \frac{V_m}{z}\sin(\omega t + \alpha - \theta) - \frac{V_m}{z}\sin(\alpha - \theta)e^{-t/\tau}$$

- At $t = t_0$, i = 0, then

$$i(t) = \frac{V_m}{z}\sin(\omega t_0 + \alpha - \theta) - \frac{V_m}{z}\underbrace{\sin(\omega t_0 + \alpha - \theta)e^{-(t-t_0)/\tau}}_{\text{(transient term)}}$$

In general, $\alpha = 0$ (Most of the numerical) if given we need to consider

For no-transients $\Rightarrow \omega t_0 - \theta = 0$ ; $\boxed{t_0 = \theta/\omega}$

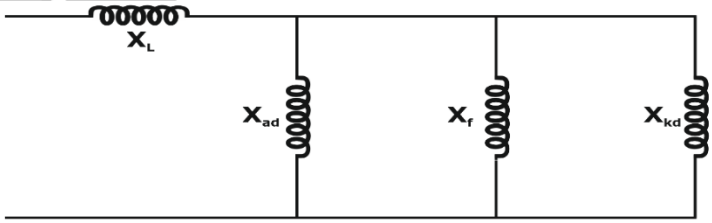
- Symmetrical RMS current (Steady state) $I_{symmetrical} = \frac{V_m}{\sqrt{2}\,|z|} = \frac{V_{rms}}{|z|}$
- Asymmetrical RMS current ($I_{asy}$): $I_{asymmetrical} = \sqrt{\left(\frac{V_m}{\sqrt{2}\,z}\right)^2 + \left(\frac{V_m}{z}\sin(\theta - \alpha)e^{-t/\tau}\right)^2}$
- Maximum Momentary current ($i_{mm}$): Maximum at $\omega t + \alpha - \theta = \pi/2$ and generally $\tau$ is high.

  so $e^{-t/\tau} \simeq 1$ ; $\therefore \quad i_{mm} = \frac{V_m}{z} - \frac{V_m}{z}\sin(\alpha - \theta)$

  $\theta \simeq 90°$, as $x_L >> R$ ; equation becomes $i_{mm} = \frac{V_m}{z}(1 + \cos\alpha)$

  If $\alpha$ = 0° in source voltage, $i_{mm} = \frac{2V_m}{z}$

- Equivalent circuit model of synchronous machine in sub-transient, transient and steady state period is analyzed by below figure: Where $X_{ad}$ ,$X_r$, $X_{kd}$ are reactances due to armature reaction, field winding and damper winding respectively. $X_L$ is the leakage reactance




Sub transient period: $X_{ad}$, $X_r$, $X_{kd}$ three reactance's will be there along with $X_L$

Transient period: $X_{ad}$,$X_r$ will be there along with $X_L$

Steady state period: only $X_{ad}$ will be there along with $X_L$

- The sequence voltage can also be expressed in terms of phase voltages and vice versa as

given in matrix form.

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} V_{a0} \\ V_{a1} \\ V_{a2} \end{bmatrix} \text{ and } \begin{bmatrix} V_{a0} \\ V_{a1} \\ V_{a2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

The above equations can be expressed also in terms of phase and sequence currents as

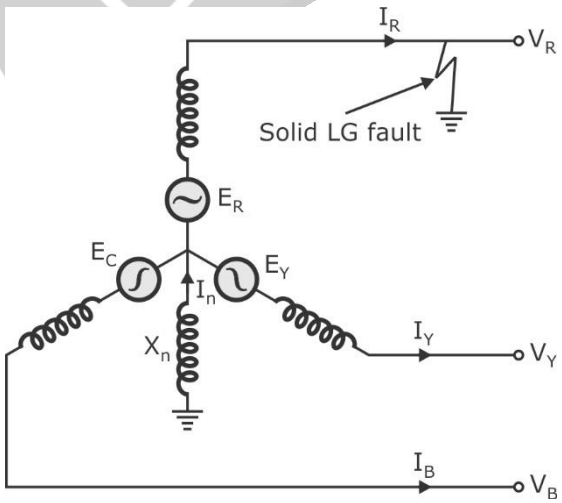
$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} I_{a0} \\ I_{a1} \\ I_{a2} \end{bmatrix} \quad \text{And} \quad \begin{bmatrix} I_{a0} \\ I_{a1} \\ I_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}$$

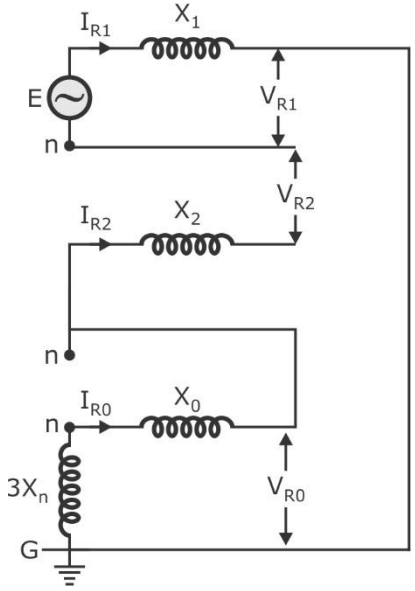
- Sequence Impedances of Transmission Lines: $Z_s$ is self-impedance and $Z_m$ is mutual impedances of transmission line then
  1. Zero-sequence impedance, i.e., $Z_0 = Z_s + 2Z_m$
  2. Positive-sequence impedance, i.e., $Z_1 = Z_s - Z_m$
  3. Negative-sequence impedance, i.e., $Z_2 = Z_s - Z_m$
     - Symmetrical Fault or 3phase fault: A three phase bolted fault describes the condition where the three conductors are physically held together with zero impedance between them, just as if they were bolted together.

       Fault current $I_f = I_{a1} = \dfrac{E_a}{Z_{1eq} + Z_f}$

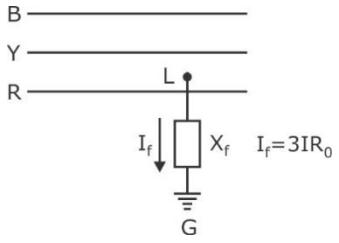
**Unsymmetrical faults:**

**(i) <u>LG Fault</u>:**



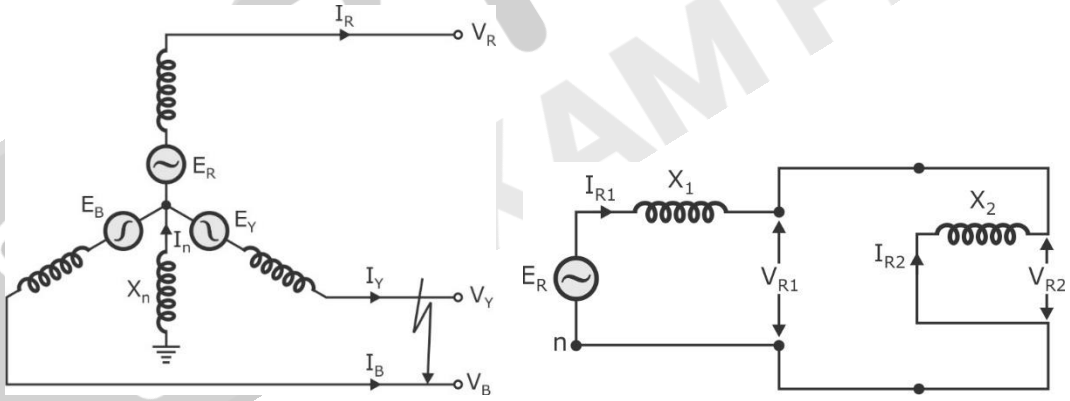
- $I_{R0} = I_{R1} = I_{R2} = \dfrac{E}{X_1 + X_2 + X_0 + 3X_n}$ and $I_f = 3I_{R0} = 3I_{R1} = 3I_{R1} = 3I_{R2}$
- **Special case:** If the LG fault takes place with fault impedance/reactance ($X_f$) as shown below. Then



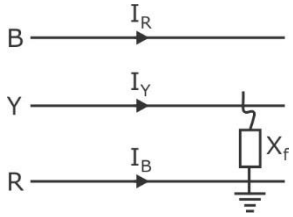

Sequence current relations will not change, but sequence voltage relations will change. $I_{R0} = I_{R1} = I_{R2} = \dfrac{E_R}{X_1 + X_2 + X_0 + 3X_n + 3X_f}$

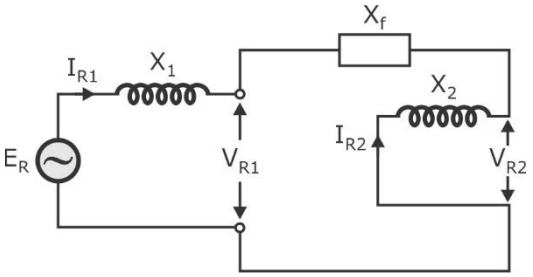
**(ii) LL fault:**

- $I_f = -j\sqrt{3}\, I_{R1}$ or $j\sqrt{3}\, I_{R2}$
- $I_{R1} = \dfrac{E}{X_1 + X_2} = -I_{R2}$



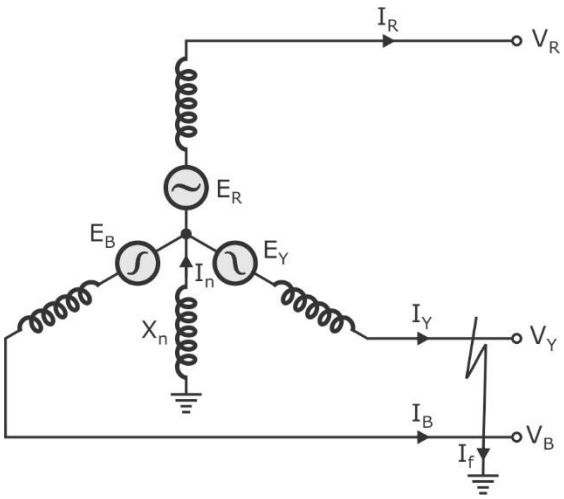

- **Special case:** If the L-L fault takes place with a fault impedance/ reactance ($X_f$) as shown in figure below.

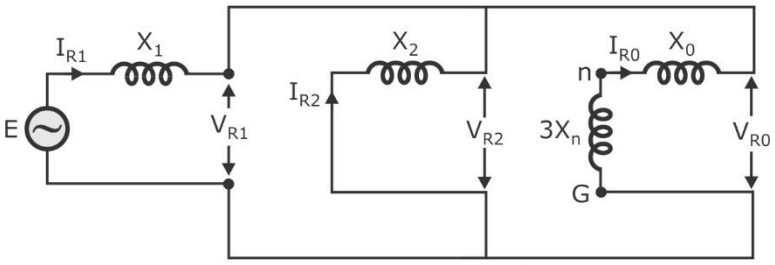



Then, $I_{R1} = -I_{R2} = \dfrac{E_R}{X_1 \times X_2 + X_f}$ and $I_f = -j\sqrt{3}\, I_{R_1}$

**(iii) LLG Fault:**




- From ohm's law,

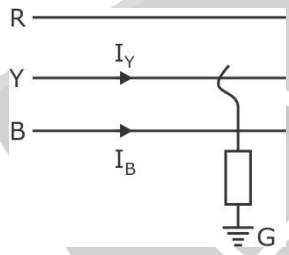
$$I_{R1} = \frac{E}{X_1 + [X_2][X_0 + 3X_n]}$$

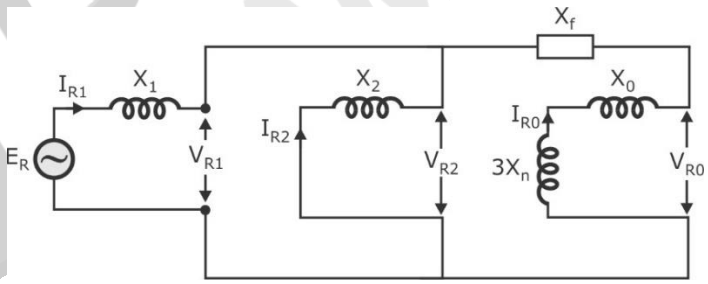
$$I_{R2} = -I_{R1} \cdot \frac{X_0 + 3X_n}{X_2 + X_0 + 3X_n}$$

$$I_{R0} = -I_{R1} \cdot \frac{X_2}{X_2 + X_0 + 3X_n}$$

$$I_f = 3I_{R0}.$$

**Special case:** If the LLG fault takes place with a fault impedance/reactance ($X_f$) as shown below.




- From ohm's law.

$$I_{R1} = \frac{E}{x_1 + [x_2 \parallel (x_0 + 3x_n + 3x_f)]}$$

$$I_{R0} = -I_{R1} \frac{x_2}{x_2 + x_0 + 3x_n + 3x_f}$$

$$I_f = I_Y + I_B = 3I_{R0}$$

**Note Points:**

- LLG fault with isolated neutral grounding is considered as LL Fault·
- LG fault with isolated grounding results to arcing grounds due to capacitance effect, to reduce that Peterson coil is employed
- Short Circuit MVA: Power delivered during fault

$$\text{SC MVA (p.u)} = E_{a1}(\text{p.u}) \times I_{a1}(\text{p.u})$$

SC MVA (Actual)= SC MVA (p.u) × BASE MVA, Use the same for all faults

- An open circuit fault result in
  (i) Increase in phase voltage
  (ii) Decrease in current
  (iii) Increase in power factor
- A short circuit fault result in
  (i) Increase in current
  (ii) Decrease in voltage
  (iii) Decrease in power factor
- LG fault is more frequent fault and is due to flashover of insulator or falling of tree branches

## 11. POWER SYSTEM STABILITY

1. **Steady state stability:** Refers to small disturbances

- Condition for stability: $\frac{\partial p_e}{\partial \delta} > 0$
- Steady state stability is improved by using bundle conductors, series capacitor, double circuit line, SVC (static VAR compensation)
- for small disturbances, the oscillations frequency is given by

$$\omega = \sqrt{\frac{\left(\frac{\partial p_e}{\partial \delta}\right)_{\delta=\delta_o}}{M}} \text{ rad/s}$$

2.**Transient Stability:** refers to large disturbance due to short circuit faults. The following are the methods to improve transient stability

- By minimizing fault severity and duration
- Increase the restoring synchronizing forces
- Reduce the accelerating torque through prime-mover control
- Reduce the accelerating torque by applying artificial load
- Earthing of transformer neutral through resistance or reactance
- Series compensation of lines

**3.Swing equation**:

$$M\frac{d^2\delta}{dt^2} = P_a = P_s - P_e$$

Where, $M = J_m\omega_m$ is the angular momentum of the rotor and is called the inertia constant.

$P_s$ is the mechanical power input to the prime mover.

$P_e$ is the electrical power output.

$P_a$ is the accelerating power which accounts for any unbalance between $P_i$ and $P_o$ in watts.

**4. Inertia Constant**:

- The kinetic energy sored in rotating parts of synchronous machine is expressed as

$$K.E = \frac{1}{2} I\omega_s^2 \times 10^{-6} \text{ MJ}$$

- Now, the inertia constant, $H = \frac{\text{Kinetic Energy stored}}{\text{Machine rating}} = \frac{\frac{1}{2} I\omega^2}{S}$

$$2HS = I.\omega^2 = M\omega \qquad (\because M = I.\omega)$$

$$M = \frac{2HS}{2\pi f} = \frac{HS}{\pi f} \text{ MJ-sec/elec-rad}$$

Here S is the three-phase rating of the machine (in MVA).

**5. Alternators in parallel:**

- Coherently then $\delta_1 = \delta_2 = \delta$

In Coherent operation, there is no real power exchange between the systems $M_{eq} = M_1 + M_2$

$$\frac{G_{eq}H_{eq}}{\pi f} = \frac{G_1H_1}{\pi f} + \frac{G_2H_2}{\pi f}$$

$$H_{eq} = \frac{G_1H_1}{G_{eq}} + \frac{G_2H_2}{G_{eq}}; \; G_{eq} \text{ is common base}$$

- In non-coherent Operation: $\delta_1 \neq \delta_2$

One machine will deliver power and other machine will absorb power i.e., one as generator and one is motor.

Here $M_{eq} = \frac{M_2P_{m1} - M_1P_{m2}}{M_1 + M_2}$ where $P_{m1}$ and $P_{m2}$ are mechanical powers of both machines

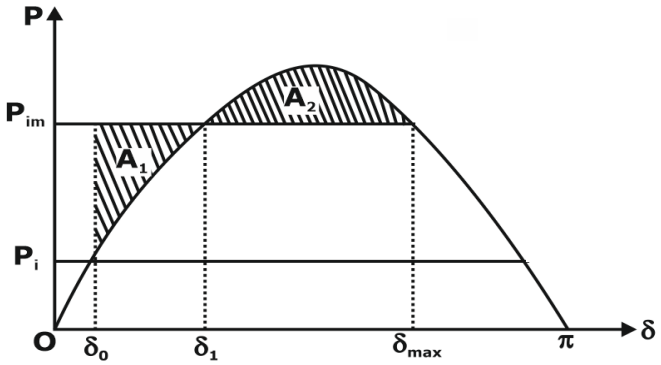
**6. Equal Area Criteria:**

Applicable only for single machine system. By using this we do the transient stability analysis without using the swing equation.

$$\Delta\omega = \left(\frac{d\delta}{dt}\right) = \sqrt{\frac{2}{M}\int_{\delta_0}^{\delta} P_a d\delta}$$

$$\Delta\omega = \frac{d\delta}{dt} = 0 \quad \text{when} \int_{\delta_0}^{\delta_m} P_a d\delta = 0$$

**(i)Sudden Increase in Power Input:**



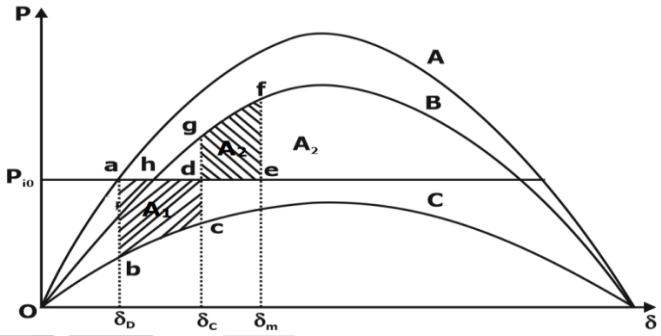

For equal area criteria $A_1 = A_2$

$$P_{im} \times (\delta_1 - \delta_0) - \int_{\delta_0}^{\delta_1} P_{max} \sin\delta \, d\delta = \int_{\delta_1}^{\delta_{max}} P_{max} \sin\delta \, d\delta - P_i (\delta_{max} - \delta_1)$$

Where $\delta_{max} = 180° - \delta_0$

Critical fault clearing time ($t_{cr}$): $t_{cr} = \sqrt{\frac{2M(\delta_{cr} - \delta_0)}{P_i}}$ ; $\delta_{cr}$= Critical clearing angle

**(ii)Three-phase Fault at the Middle of One Line of the Double Circuit Line**:




Critical clearing angle can be calculated by the given formula

$$\cos\delta_{cr} = \frac{P_{io}(\delta_{max} - \delta_o) + P_{max\,B}\cos\delta_{max} - P_{max\,C}\cos\delta_o}{P_{max\,B} - P_{max\,C}}$$

## 12. LOAD FLOW STUDIES

| Bus | Known | Unknown |
|---|---|---|
| Slack bus/swing bus/ reference bus | $\|V\|, \delta$<br>$V\angle\delta = 1\angle 0$ | P, Q |
| Generator Bus/PV Bus | P, V | Q,$\delta$ Unknown<br>$Q_{g,min} < Q_g < Q_{g,max}$ if it is out of limits PV bus is considered as PQ bus |

| | | |
|---|---|---|
| Load Bus/PQ bus | P, Q | $\|V\|,\delta$ |

**Important Points:**

**1)** PQ Bus: Load Bus

Fixed Capacitor bus/ Fixed Inductor Bus

**2)** PV Bus : Damper Bus

Reactive power support Bus

Voltage controlled Bus/ Variable shunt capacitor Bus

Variable capacitor/Inductor Bus

Generator Bus

- Size of Jacobian Matrix: (2PQ+PV) X (2PQ+PV)
- Slack Bus is nothing but one of the generator Bus which can supply the line losses totally by sharing the demand also

**Z-Bus Construction:**

- By means of an impedance $Z_b$, a new bus is added to the reference bus then the order of z-bus matrix is increased by one by adding column and row of zeros with the diagonal element $Z_b$
- If a new bus is added to the existing bus K then order will increase by one by adding a row and column identical to row k and column k and diagonal element is $Z_{kk}+ Z_b$
- If impedance $Z_b$ is added between reference bus and existing $k^{th}$ bus then new Z-bus matrix is

$$\left[Z^{new}\right] = \left[Z^{old}\right] - XX^T\beta$$

X= $K^{th}$ Column matrix

$$\beta = \frac{1}{Z_{kk} + Z_b}$$

- If impedance is added between two existing bus(j and k)

$$\left[Z^{new}\right] = \left[Z^{old}\right] - XX^T\beta$$

X = $j^{th}$ column - $k^{th}$Column matrix

$$\beta = \frac{1}{Z_{kk} + Z_{jj} - 2Z_{jk} + Z_b}$$

**Y-BUS:**

1) Symmetric matrix(means upper and lower triangle matrices are same)
2) Sum of admittances connected to that specific bus will be the diagonal elements of Y-Bus
3) Negative of admittance connected between any two bus gives off diagonal elements of Y-Bus
4) Sparse Matrix (Most elements are Zero):

$$\%\ \text{Sparsity} = \frac{\text{Number of zero elements}}{\text{Total number of elements}}$$

Number of Non-Zero elements = (Number of transmission lines) x2 + No. of Bus

5) Both Y-bus and Z-bus are symmetrical in nature

Comparison of load flow methods:

| **Newton Raphson Method** | **Gauss –seidel method** |
|---|---|
| Time taken for each iteration is high | Time taken for each iteration is less |
| Convergence is independent on choice of slack bus | Convergence depends on choice of slack bus |
| Applicable for larger power systems | Applicable for small power systems |

## 13. POWER SYSTEM PROTECTION

- Restriking voltage in a circuit is given by $e_{RV}=E_m\left(1-\cos\left(\frac{t}{\sqrt{LC}}\right)\right)$
- Rate of rise of recovery voltage RRRV = $\frac{E_m}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)$
- The average RRRV = $\frac{2E_m}{\pi\sqrt{LC}}$
- Prospective voltage develop across circuit breaker during current chopping is $v = i\sqrt{\frac{L}{C}}$
- The value of resistor required to be connected across the circuit breaker contacts which will give no transient oscillation (critical damping), is R = $0.5\sqrt{\frac{L}{C}}$

Where L, C are the inductance and capacitance up to the circuit breaker

- Natural frequency of oscillations, $f_n = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$
- Frequency of damped oscillation, f = $\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{1}{4R^2C^2}}$
- Symmetrical breaking current = rms value of ac component = $\frac{X}{\sqrt{2}}$
- Asymmetrical breaking current = rms value of total current = $\sqrt{\left(\frac{X}{\sqrt{2}}\right)^2+Y^2}$

Where X = maximum value of ac component

Y = dc component

- Breaking capacity = $\sqrt{3}\times V\times I\times 10^{-6}$ MVA
- Making capacity = 2.55 × symmetrical breaking capacity.

- The universal relay torque equation is given as follows $T = K_1 I^2 + K_2 V^2 + K_3 VI (\theta - \tau) + K$

**Important note points in protection:**

- Isolator is a no load (off load) switch.
- Electromagnetic attraction relays are used for both ac and dc.
- Electromagnetic relay has limited application for ac protection because pulsating force produce in ac application may damage the relay contact.
- Induction relays are used only for ac.
- For an impedance relay to operate**:**

  $\left|\frac{V}{I}\right| < \left|Z_{relaysetting}\right|$ . Impedance relay is used for protection of medium transmission lines
- For a reactance relay to operate:

  $|X| <$ constant value .

  Reactance relay used for short transmission lines
- For a mho relay to operate:

  $\left|\frac{V}{I}\right| < \left|K\cos(\theta - \tau)\right|$

  $\tau$ is maximum torque angle and $\theta$ is impedance angle

  Mho relay is used for long transmission lines protection, and it is least affected by power surges
- Over current relay operates when $I_{coil} > I_{pickup}$
- Current Setting (C.S): It is defined as ratio of pickup current to the relay current setting.

$$C.S = \frac{I_{pickup}}{I_{relay}} \times 100$$

- Plug setting multiplier (PSM) = $\frac{\text{Secondary C.T current}}{\text{Pickup current}} = \frac{\text{fault current}}{\text{CT ratio} \times \text{Pickup current}}$

  PSM>1: Relay operates

  PSM<1: Relay does not operate

  PSM=1: Relay on threshold
- Actual relay operating time = Time corresponding to PSM× Time setting multiplier (TMS)
- A relay is generally provided with control to adjust the time of operating this adjustment is known as TMS.
- Instantaneous over current relay operating time ($t_{op}$) = 0.1 sec.
- Definite time overcurrent relay $t_{op}$ = (TMS) × $t_{rating}$
- IDMT over current relay $t_{op} = \frac{0.14 \times TMS}{PSM^{0.02} - 1}$
- Very inverse over current relay $t_{op} = \frac{13.5 \times TMS}{PSM - 1}$

- Extremely inverse OC relay $t_{op} = \frac{80 \times TMS}{PSM^2 - 1}$
- Buchholz relays have the capability of anticipating the possible major fault in transformer.

- CT configurations:

| | Power transformer connections | CT's connects in differential protection |
|---|---|---|
| 1. | Y/ Δ | Δ/Y |
| 2. | Δ /Y | Y/ Δ |
| 3. | Y/Y | Δ / Δ |
| 4. | Δ / Δ | Y/Y |

- Different protection schemes:

| | |
|---|---|
| Rotor protection | Rotor earth fault relay |
| Unbalanced loading | Negative sequence |
| Over speed protection | Watt metric relay |
| Failure of prime mover | Watt metric relay with directional characteristics |
| Loss of excitation | Offset mho Split phase relaying relay. |
| Stator protection | Differential relay |
| Interturn faults | Split phase relaying |
| Phase to phase and phase to ground fault | Differential protection phase to ground fault |
| Power transformer, Alternator | Differential relaying |
| Transmission lines | Distance relaying |

- Biased differential relay operates when

$$r|I_1 - I_2| - r\frac{N_r}{N_o}\left|\frac{I_1 + I_2}{2}\right| > |I_{pickup}|$$

r = CT ratio

$N_r$ = number of turns in restraining coil

$N_o$ = number of turns in operating coil

$\frac{N_r}{N_o}$ = Bias of differential relay

$$\% \text{ Bias} = \frac{i_1 - i_2}{\frac{i_1 + i_2}{2}} \times 100$$

***

BYJU'S EXAM PREP

# Electrical Engineering

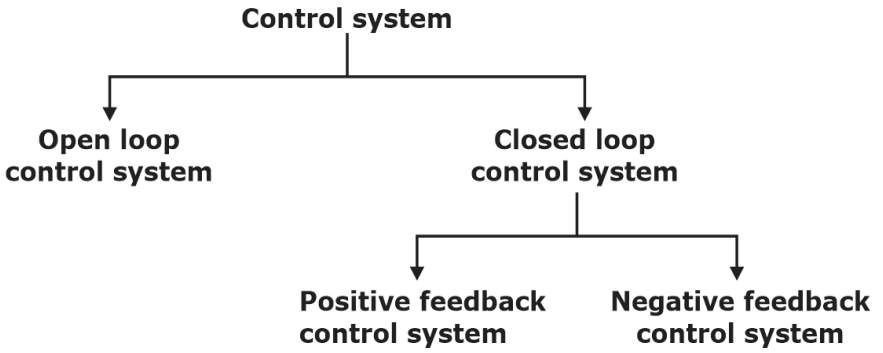
## Control Systems

SHORT NOTES

# CONTROL SYSTEMS(FORMULA NOTES/SHORT NOTES)

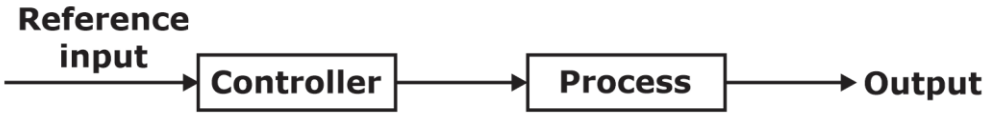
## INTRODUCTION

**TYPES OF CONTROL SYSTEMS**



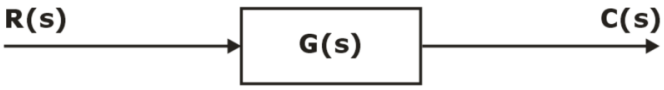

**1. Open loop control system:**

An open loop control system also called a non-feedback control system.



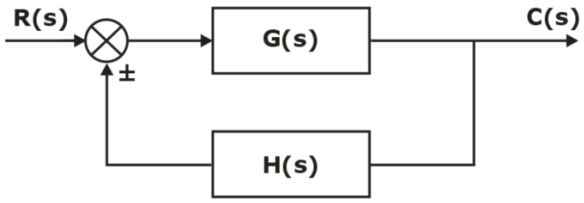

- In open loop-controlled system output is controlled.
- Accuracy of open loop control system depends on the accuracy of input calibration.

Block Diagram Representation of Open Loop System:




$$G(s) = \frac{C(s)}{R(s)}$$

**2. Closed Loop Control System:**




$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)H(s)}$$

**Comparison Between Open Loop & Close Loop Control System**

| S.No. | Open loop control system | Close loop control system |
|---|---|---|
| 1. | The changes in output of system cannot be corrected. | The changes in output of system can be corrected. |
| 2. | No sensor is available. | Sensor is available. |
| 3. | It is non feedback control system. | It is feedback control system. |
| 4. | The accuracy of an open-loop system depends on the calibration of the input. Any departure from pre- determined calibration affects the output. | The close loop system works more accurately since the error between the reference input and the output is continuously measured through feedback. |
| 5. | Generally stable but cannot be stabilize if become unstable. | Due to feedback, system can become stable and can be stabilized. |
| 6. | More sensitive for parameter variation. | Less sensitive for parameter variation. |
| 7. | Simple & Economic | Complex & Costly |

## Transfer Function

The transfer function of an LTI systems may be defined as the ratio of Laplace transform of output to the Laplace transform of input under the assumption.

$$G(s) = \frac{Y(s)}{X(s)}$$

•The transfer function is completely specified in terms of its poles and zeros and the gain factor.

•The TF function of a system depends on its elements, assuming initial conditions as zero and is independent of the input function.

•To find DC gain of the system, put s = 0.

Example:

$$G(s) = \frac{s+4}{s^2+6s+9}; \quad \text{Gain} = \frac{4}{9}$$

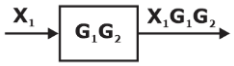
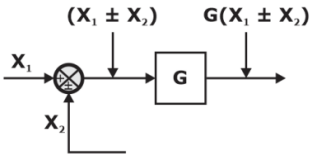
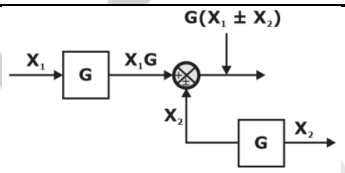
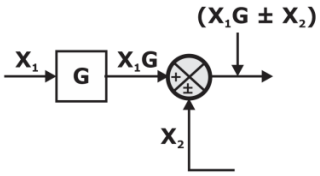
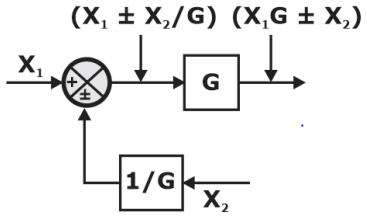
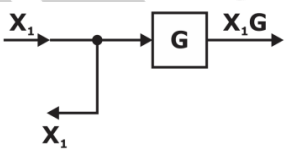
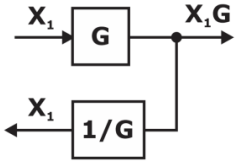
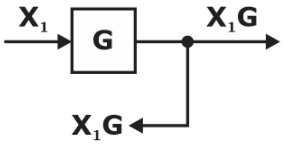
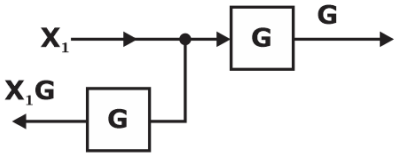
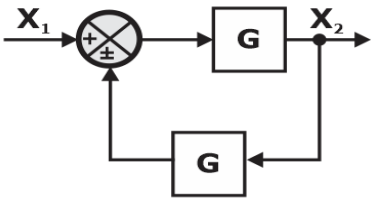
If a step, ramp or parabolic response of Transfer Function is given, then we can find impulse response directly through differentiation of that Transfer Function.

$\frac{d}{dt}$ (Parabolic response) = Ramp response

$\frac{d}{dt}$ (Ramp response) = Step response

$\frac{d}{dt}$ (Step response) = Impulse response

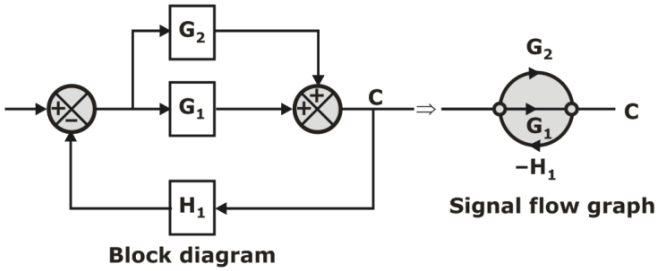
**Rules of Block Diagram:**

| Rule | Original diagram | Equivalent diagram |
|---|---|---|
| 1. Combining blocks in cascade | |  |
| 2. Moving a summing point after a block |  |  |
| 3. Moving a summing point ahead of a block |  |  |
| 4. Moving a take off point after a block |  |  |
| 5. Moving a takeoff point ahead of a block |  |  |
| 6. Eliminating a feedback loop |  | |

**Signal Flow Graphs:**

It is a graphical technique that deals with the relation between the variable of a system described in the form of set of linear algebraic equation.

•Signal flow graph of Block diagram.




**Mason's formula:**

This is a technique for reducing signal flow graphs to single transfer functions that relate the output of a system to its input. There is general gain formula, called **Mason's rule**, that allow the determination of the input-output relations of an SFG by inspection. The transfer function of a system is given as,

$$T.F = \frac{C(s)}{R(s)} = \frac{\sum_{k=1}^{N} P_k \Delta_k}{\Delta}$$

Where,

K= 1, 2, 3, .....

N = no of forward path

$P_K$ = Gain of $K^{th}$ forward path

$\Delta_K$ = 1 − [Sum of individual loop gain not touching to $K^{th}$ forward path] + [Sum of product of 2 non touching loop gain that not touching to $k^{th}$ forward path]

Δ = Determinant of the graph or characteristics function

Δ = 1 − (Sum of all individual loop gain) + (Sum of gain products of all possible combinations of two non-touching loops) − (Sum of the gain products of all possible combinations of three non-touching loops) + ……

## Time Response Analysis of Control System

The time response of a control system is divided in two parts:

1. Transient response
2. Steady state response

**Transient response** is dependent upon the system poles only and not on the type of input. It is due to the non-zero initial conditions. It is therefore enough to analyze the transient response using a step input.

**The steady-state response** depends on system dynamics and the input signal. It is then examined using different test signals by final value theorem.

Total response of a system = Transient response + Steady state response

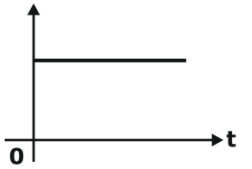
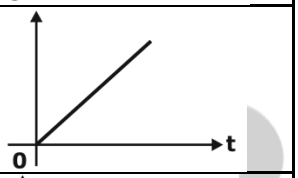
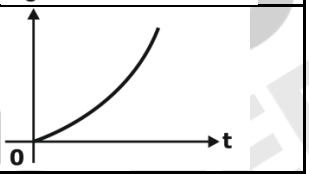
$C(t) = C_{tr}(t) + C_{ss}(t)$

Where, C(t) = overall response of the system,

$C_{tr}(t)$ = transient response component of the system and

$C_{ss}$ (t) = steady state response component of the system

**Different test signal and their transforms**

| Test signal | Time domain | Laplace Transform | Wave form |
|---|---|---|---|
| Unit impulse | δ (t) | 1 | |
| Unit step function | u(t) | 1/s |  |
| Unit Ramp | tu(t) | $1/s^2$ |  |
| Unit parabolic | $\frac{t^2}{2}u(t)$ | $1/s^3$ |  |

$\int_{-\infty}^{t} \text{Impulse}\, dt = \text{step signal}$ ; $\frac{d}{dt}\text{Ramp} = \text{step}$ ;

$\int_{-\infty}^{t} \text{Ramp}\, dt = \text{parabolic}$ ; $\frac{d}{dt}\text{Parabolic} = \text{Ramp}$

$\int_{-\infty}^{t} \text{step}\, dt = \text{Ramp}$; $\frac{dt}{dt}\text{step} = \text{Impulse}$

## INITIAL VALUE AND FINAL VALUE THEOREM

1. **Initial value of the response:**

$$c(0) = \lim_{t\to 0} c(t) = \lim_{s\to\infty} sC(s)$$

Initial value theorem is applied only when the number of poles of C(s) is more than the number of zeros i.e. function C(s) must be strictly proper.

2. **Final value of the Response:**

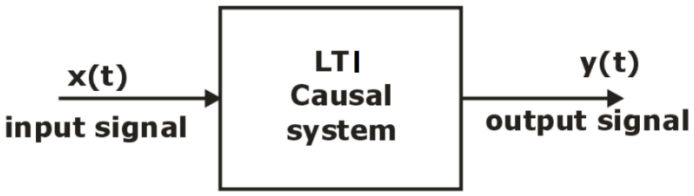
$$c(\infty) = \lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s)$$

Final value theorem is applied when all the poles of C(s) lie in the left half of the s-plane.

## TIME RESPONSE OF SECOND ORDER CONTROL SYSTEM

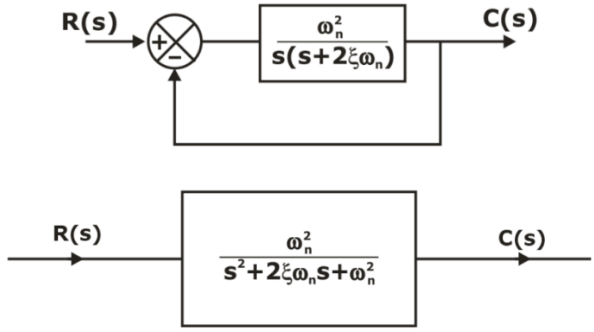
A second order control system is one for which the highest power 's' in the denominator of its transfer function is equal to 2.

Consider a linear time-invariant system where input to the system is given by x(t) & its corresponding output as y(t).




The block diagram representation in Laplace domain is

or




**Block diagram of second order control system**

The characteristic equation of the second order system is

$$\text{C.E.} = s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

**1. Case 1: ξ = 0**

In this case, the poles of the system lie on jω axis and hence the system is said to be undamped system.

**2. Case 2: ξ = 1**

In this case, the roots or poles of the system is always real and equal and equals to $-\omega_n$ i.e.

$$s_1, s_2 = -\omega_n$$

Therefore, the system is said to be critically damped system.

**3. Case 3: 0 < ξ < 1**

In this case, the poles of the system are complex conjugate to each other. i.e.

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$

Hence, the system is called under damped system.

**4. Case 4: ξ > 1**

In this case, the roots or poles of the system are real but unequal i.e.,

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Therefore, the system is said to be over damped system.

**For Step input:**

- $$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\,t \pm \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$
- $$e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\,t \pm \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$
- $$e_{ss} = \lim_{t\to\infty} \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\,t \pm \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$

  $\rightarrow \xi \rightarrow$ Damping ratio; $\xi\omega_n \rightarrow$ Damping factor

1. $\xi < 1$ **(Under Damped):**

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_d t \pm \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$

2. $\xi = 0$ **(Undamped):**

$$C(t) = (1 - \cos\omega_n t)$$

3. $\xi = 1$ **(Critically Damped):**

$$C(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

4. $\xi > 1$ **(overdamped):**

$$C(t) = 1 - \frac{e^{-\left(\xi - \sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)}$$

**Time constant of the response**

$$T = \frac{1}{\left(\xi - \sqrt{\xi^2-1}\right)\omega_n}$$

$T_{undamped} > T_{overdamped} > T_{underdamped} > T_{critical\ damp}$

**Time Domain Specifications:**

- **Rise time ($t_r$):**

Rise time is defined as the time at which the output reaches its final value for the first time.

$$t_r = \frac{\pi - \phi}{\omega_n \sqrt{1 - \xi^2}}; \qquad \phi = \cos^{-1} \xi$$

- **Peak time ($t_p$):**

Peak time is defined as the time at which the response of the system reaches its peak value for the first time.

$$t_p = \frac{n\pi}{\omega_d}$$

For first peak, n = 1 (maxima)

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

- **Peak Overshoot ($M_p$):**

Peak overshoot is defined as the deviation of the response of the system from its final value at its first time. $M_p$ can be positive or negative.

$$\% M_P = e^{-\xi\omega_n/\sqrt{1-\xi^2}} \times 100$$

- **Settling time ($t_s$):**

Settling time is the time required sinusoidal amplitude settled to its final value.

$t_s$ = 3T for 5% tolerance,

$t_s$ = 4T for 2% tolerance

- **Delay time ($t_d$):**

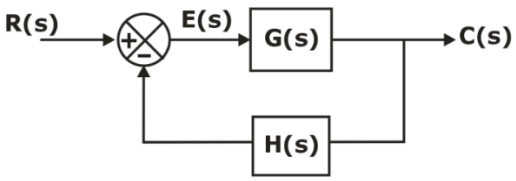
Delay time is defined as the time required to reach half of its final value from zero instant.

$$t_d = \frac{1 + 0.7\xi}{\omega_n}$$

**STEADY STATE ERROR ($E_{ss}$)**

Steady state error is defined as the amount of output deviated from its desired response during steady state is known as steady state error. It is denoted by $e_{ss}$.

Now, consider a closed loop system,




C(s) = E(s)G(s)

Error signal is given as

$$E(s) = \frac{C(s)}{G(s)}$$

For closed loop system,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$C(s) = \frac{G(s).R(s)}{1 + G(s)H(s)}$$

Putting the value of C(s) in E(s)

$$\boxed{E(s) = \frac{1}{1 + G(s).H(s)}.R(s)}$$

Where E(s) is error signal and R(s) is input signal.

G(s) H(s) is the open loop transfer function.

Using final value theorem of Laplace theorem,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e_{ss} = \lim_{s\to 0}\left\{ s\left[\frac{1}{1+G(s)H(s)}\right]R(s)\right\}$$

**Static Error Coefficients:**

- **Step input:**

  $$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{sR(s)}{1 + GH}$$

  $$e_{ss} = \frac{1}{1 + K_P} \text{ (Positional error)}$$

  $$K_P = \lim_{s\to 0} G(s)H(s)$$

  Where, $K_P$ is static position error constant.

- **Ramp input:**

  $$e_{ss} = \frac{1}{K_v}$$

  $$K_v = \lim_{s\to 0} s\, G(s)H(s)$$

  Where, $K_V$ is static velocity error constant.

- **Parabolic input:**

  $$e_{ss} = \frac{1}{K_a}, \qquad K_a = \lim_{s\to 0} s^2\, G(s)H(s)$$

Where, $K_a$ is static acceleration error constant.

Type < input $\rightarrow e_{ss} = \infty$

Type = input $\rightarrow e_{ss}$ (finite value)

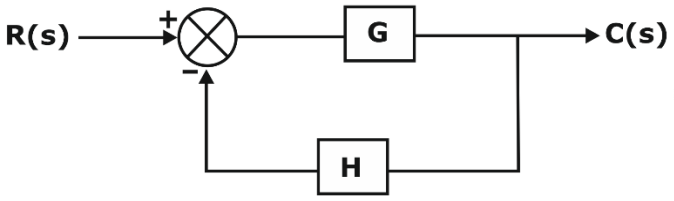
Type > input $\rightarrow e_{ss} = 0$

## Steady state Error for Different types of system:

| Type of System | Unit Step Input | Unit Ramp Input | Unit Parabolic Input |
|---|---|---|---|
| Type – 0 System | $K_p = K$<br>$e_{ss} = \frac{1}{1+K}$ | $K_V = 0$<br>$e_{ss} = \infty$ | $K_a = 0$<br>$e_{ss} = \infty$ |
| Type – 1 System | $K_p = \infty$<br>$e_{ss} = 0$ | $K_V = K$<br>$e_{ss} = \frac{1}{K}$ | $K_a = 0$<br>$e_{ss} = \infty$ |
| Type – 2 System | $K_p = \infty$<br>$e_{ss} = 0$ | $K_V = \infty$<br>$e_{ss} = 0$ | $K_a = K$<br>$e_{ss} = \frac{1}{K}$ |

# Routh Hurwitz Stability Criteria

$$CLTF = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$




1 + G(s)H(s)= 0 is called as a characteristic equation.

**FORMATION OF ROUTH ARRAY**

For an $n^{th}$ order system, the characteristic equation is

$$q(s) = A_0 s^n + A_1 s^{n-1} + .... + A_{n-1} s + A_n = 0$$

Routh Hurwitz Criterion is based on ordering the coefficient of the system's characteristic equation into an array called the **Routh array.**

The first two rows of the Routh array is filled by the coefficient of the characteristic equation.

If n is odd, then the first row is filled up by all the odd positioned coefficients and the second row is filled up by all even positioned coefficient and vice versa for n even.

Considering n is even

| | | |
|---|---|---|
| $s^n$<br>$s^{n-1}$ | $A_0$ $A_2$ $A_4$<br>$A_1$ $A_3$ $A_5$ | |

The further rows are evaluated as

| | | | | |
|---|---|---|---|---|
| $S^n$ | $A_0$ | $A_2$ | $A_4$ | $A_6$ |
| $S^{n-1}$ | $A_1$ | $A_3$ | $A_5$ | |
| $S^{n-2}$ | $B_1$ | $B_2$ | $B_3$ | |
| $S^{n-3}$ | $C_1$ | $C_2$ | $C_3$ | |
| \| | \| | | | |
| $S^0$ | $A_n$ | | | |

$$B_1 = \frac{A_1 A_2 - A_0 A_3}{A_1}$$

$$B_2 = \frac{A_1 A_4 - A_0 A_5}{A_1}$$

$$C_1 = \frac{B_1A_3 - A_1B_2}{B_1}$$

$$C_2 = \frac{B_1A_5 - A_1B_3}{B_1}$$

The process continues till the last row of the array contains only the independent coefficient of the characteristic equation.

## STABILITY RULES WITH RESPECT TO ROUTH MATRIX

- The closed loop system is said to be stable if all the elements in the first column of the Routh matrix/array have same sign(either positive or negative).
- A system is said to be unstable if there is at least one sign change in the first column of the Routh array.
- Number of sign changes in the first column of the Routh's array indicate the number of closed loop poles in the right half of the s-plane.
- If a complete row in a Routh array becomes zero, then the system is said to be having symmetric poles about the origin.
- Routh Hurwitz Criterion cannot be applied for the system with exponential trigonometric or imaginary coefficient.

**Note :** Routh Hurwitz Criterion gives only information about the number of roots in the right half of s-plane. It gives no information about location of the poles.

**Example :**

Find the stability of a system represented by the character equation.

$$A_0s^3 + A_1s^2 + A_2s + A_3 = 0$$

| | | |
|---|---|---|
| $s^3$ | $A_0$ | $A_2$ |
| $s^2$ | $A_1$ | $A_3$ |
| $s^1$ | $\frac{A_1A_2 - A_0A_3}{A_1}$ | |
| $s^0$ | $A_3$ | |

Solution:

For stability,

$$A_0 > 0,\ A_1 > 0,\ A_3 > 0$$

$$\frac{A_1A_2 - A_0A_3}{A_1} > 0$$

So, $A_1A_2 - A_0A_3 > 0$

$$\Rightarrow \boxed{A_1A_2 > A_0A_3}$$

$$\begin{bmatrix}\text{Internal}\\ \text{Product}\end{bmatrix} > \begin{bmatrix}\text{External}\\ \text{Product}\end{bmatrix}$$

# Root Locus

## ANGLE & MAGNITUDE CONDITION FOR ROOT LOCUS

For studying the close loop transfer function stability, let us consider a system with open loop transfer function as G(s).H(s)

For negative feedback close loop system, characteristics equation is given as,

$1 + G(s)\,H(s) = 0$

$G(s)\,H(s) = -1$

$= -1 + j0$

**Magnitude:**

$|G(s)\,H(s)| = 1$

**Angle:**

$\angle G(s)\,H(s) = \tan^{-1} b/a$

$= \tan^{-1} 0/-1$

$= \pm 180°$

## RULES OF DRAWING THE ROOT LOCUS

**1.** Root locus start from open loop poles with k = 0 and terminates at open loop zeros with k= $\infty$.

**2.** Root locus is symmetrical about real axis. It is because complex poles always exist in conjugation.

**3.** The number of branches of root locus = P if P > Z; Z if Z > P

Where, P = number of open loop poles & Z = number of open loop zeros.

**4.** Root locus exist on real axis, if number of open loop poles and zeros to the right side of a point on the real axis is an odd number.

Let,

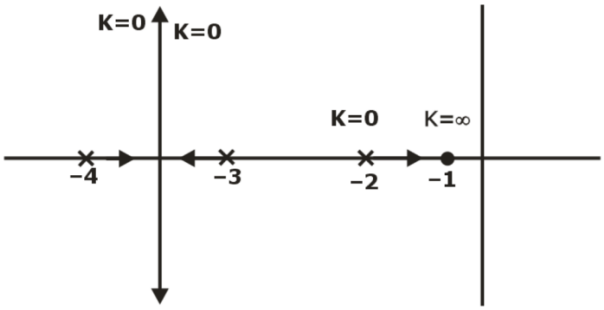
$$G(s)H(s) = \frac{s+1}{(s+2)(s+3)(s+4)}$$

Here, Number of zeros = 1 (s = -1)

Number of poles = 3 (s = -2, -3, -4)

Number of root loci/ branches = 3

Number of loci/ branches ending at infinity = P - Z = 3 – 1 = 2




### 5. Number of Asymptotes

It is the path along which the root locus moves towards infinity. The number of asymptotes is

= P - Z; if P > Z

= Z - P; if Z > P

Where, P = number of open loop poles

Z = number of open loop zeros

### 6. Centroid

It is the point of intersection of asymptotes to the negative real axis.

$$\sigma = \frac{\sum(\text{Real part of open loop poles}) - \sum(\text{Real part of open loop zeros})}{P-Z}$$

### 7. Angle of asymptotes

It is the angle made by the asymptotes with positive real axis in s-plane. Mathematically it is given by,

$$\theta_A = \frac{(2q+1)180°}{P-Z}$$

Where, q = 0, 1, 2……., (n-m-1)

Number of zeros at infinity = P - Z; only when P > Z

Number of poles at infinity = Z - P; only when Z > P,

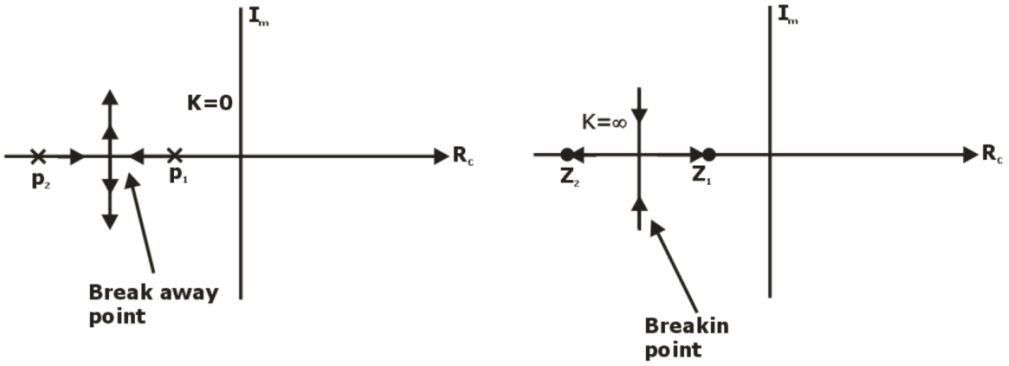
Where n is the number of finite poles & m is the number of finite zeros.

### 8. Break-away and Break-in points

These are the points at which two loci/branches coming towards each other, meet and get diverted in opposite direction.

- If there are two consecutive open loop poles on real axis and there exist a root locus between them then there is always a break away point between them.
- If there are two consecutive open loop zeros on real axis and root locus exist between them there has to be a break in point between them.

Break away and break in point lie on real axis but sometimes they may lie on complex plane also.




Breakaway and break in point can be determined by putting,

$$\frac{dK}{ds} = 0$$

### 9. Angle of Departure & Angle of Arrival

#### 9.1. Angle of Departure

It is the angle made by branch of root locus with real axis when the branch emerges from a complex pole. Angle of departure from complex open pole at s= $s_1$ is given by,

$$\phi_D\big|_{s=s_1} = 180^o + \angle GH'\big|_{s=s_1}$$

Where, GH'= (s-$s_1$) G(s)H(s)

#### 9.2. Angle of Arrival

It is the angle made by root locus with real axis when the branches of the root locus arrive at a complex zero.

Angle of arrival at a complex zero at s= $s_1$ is given by,

$$\phi_A\big|_{s=s_1} = 180^o - \angle GH'\big|_{s=s_1}$$

Where, GH'= (s-$s_1$) G(s)H(s)

### 10. Point of intersection of the RLD with respect to the imaginary axis:

It is the point of intersection of root locus with jω axis. jω cross over gives the critical value of gain parameter K at which the system becomes oscillatory. It also gives the frequency of oscillation at this critical value of K. jω cross over can be obtained by using Routh's array.

## Frequency Domain Analysis

**FREQUENCY DOMAIN SPECIFICATIONS**

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are:

### 1. Resonant Peak ($M_r$)

The maximum value of the magnitude of closed loop transfer function is called the resonant peak, $M_r$. A large resonant peak corresponds to a large overshoot in transient response.

### 2. Resonant Frequency ($\omega_r$)

The frequency at which the resonant peak occurs is called resonant frequency, $\omega_r$ This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

### 3. Bandwidth ($\omega_b$)

The Bandwidth is the range of frequencies for which normalized gain of the system is reduce by 3 db. The frequency at which the gain is reduce by 3 db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency.

The Bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

### 4. Cut-off Rate

The slope of the log-magnitude curve near the cut off frequency is called cut-off rate. The cut -off rate indicates the ability of the system to distinguish the signal from noise.

### 5. Gain Margin, $K_g$

- The gain margin, $K_g$ is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability.
- The gain margin, $K_g$ is given by the reciprocal of the magnitude of gain of open loop transfer function at phase cross over frequency.
- The frequency at which the phase of open loop transfer function is 180° is called the phase cross-over frequency, $\omega_{pc}$.

$$\text{Gain Margin}, K_g = \frac{1}{|G(j\omega_{pc})|}$$

The gain margin in dB can be expressed as,

$$K_g \text{ in dB} = 20\log K_g = 20\log\frac{1}{|G(j\omega_{pc})|}$$

### 6. Phase Margin ( $\gamma$ )

- The phase margin $\gamma$ is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability.
- The gain cross over frequency $\omega_{gc}$ is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the dB magnitude is zero).

- The phase margin $\gamma$, is obtained by adding 180° to the phase angle ϕ of the open loop transfer function at the gain cross over frequency.

Phase margin, $\gamma = 180° + \phi_{gc}$

Where, $\phi_{gc} = \angle G(j\omega_{gc})$

## FREQUENCY DOMAIN SPECIFICATIONS OF SECOND ORDER SYSTEM

- **Resonant Peak ($M_r$)**

Resonant peak, $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$

- **Resonant frequency ($\omega_r$)**

The resonant frequency, $\omega_r = \omega_n\sqrt{1-2\xi^2}$

- **BANDWIDTH ($\omega_b$):**

$$\omega_b = \omega_n\left[1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}\right]^{\frac{1}{2}}$$

## GAIN CROSSOVER FREQUENCY & PHASE CROSSOVER FREQUENCY

### 1. Gain Crossover frequency:

The gain cross-over frequency is the frequency at which the magnitude of G(jω)H(jω) is equal to unity i.e.

$$\left|G(j\omega)H(j\omega)\right| = 1$$

### 2. Phase Crossover frequency:

The phase cross-over frequency is the frequency at which the phase angle of G(jω)H(jω) is -180º i.e.

$$\angle G(j\omega)H(j\omega) = -180°$$

## GAIN MARGIN (GM) & PHASE MARGIN (PM)

### 1. Gain Margin

Gain Margin is the amount of gain in dB that can be added to the loop before closed loop system becomes unstable. The GM is given by

$$GM = 20\log_{10}\frac{1}{a} = 20\log_{10}\frac{1}{\left|G(j\omega)H(j\omega)\right|}$$

$$= -20\log_{10}\left|G(j\omega)H(j\omega)\right| \text{ at phase cross-over frequency}$$

$$= -20\log_{10}\left|G(j\omega_{pc})H(j\omega_{pc})\right|$$

**2. Phase Margin**

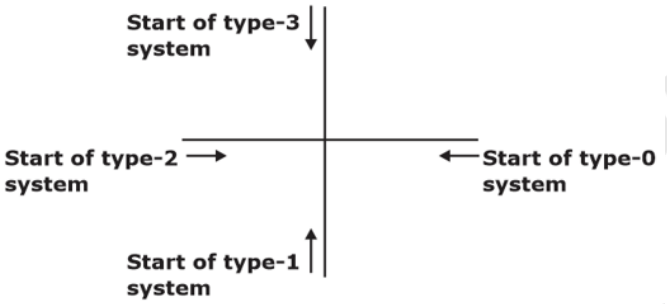
Phase Margin is the amount of pure phase delay that can be added before the system becomes unstable. The PM is given by

$$PM = 180° + \angle G(j\omega)H(j\omega) \text{ at gain cross-over frequency}$$

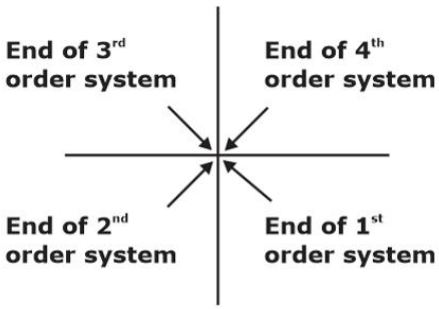
$$= 180° + \angle G(j\omega_{gc})H(j\omega_{gc})$$

## POLAR PLOT

Starting of polar plot of all minimum Phase system.




End of polar plot of all minimum Phase system.



## TABLE : Typical Sketches of Polar Plot

**Type: 0, Order: 1**

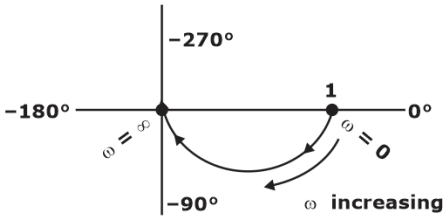
$$G(s) = \frac{1}{1+sT}$$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\angle \tan^{-1}\omega T}$$

$$= \frac{1}{\sqrt{1+\omega^2T^2}}\angle -\tan^{-1}\omega T$$

As $\omega \to 0$, $G(j\omega) \to 1\angle 0°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90°$




**Type: 1, Order: 2**

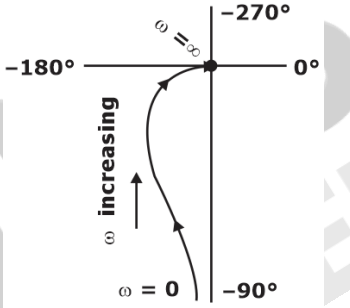
$$G(s) = \frac{1}{s(1+sT)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\angle 90°\sqrt{1+\omega^2T^2}\angle \tan^{-1}\omega T}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2T^2}}\angle -90° - \tan^{-1}\omega T$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -180°$




**Type: 0, Order: 2**

$$G(s) = \frac{1}{(1+sT_2)(1+s_2)}$$

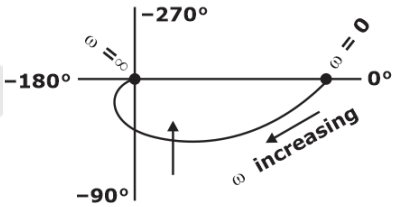
$$G(j\omega) = \frac{1}{(1+j\omega T_2)(1+j\omega T_2)}$$

$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}\angle\left(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2\right)$$

As $\omega \to 0$, $G(j\omega) \to 1\angle 0°$

As $\omega \to \infty$, $G(j\omega) \to$ -180°




**Type: 0, Order: 3**

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

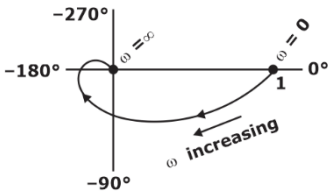
$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1\sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2\sqrt{1+\omega^2T_3^2}\angle \tan^{-1}\omega T_3}$$

$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}\angle\left(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3\right)$$

As $\omega \to 0$, $G(j\omega) \to 1\angle 0°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -270°$



**Type: 1, Order: 3**

$$G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$$

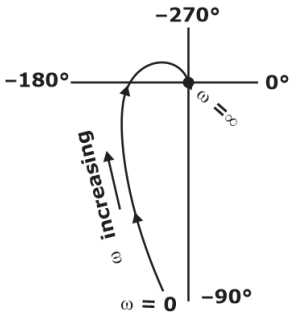
$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{1}{\omega\angle 90°\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1\sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}\angle(-90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -270°$




**Type: 2 Order: 4**

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$$

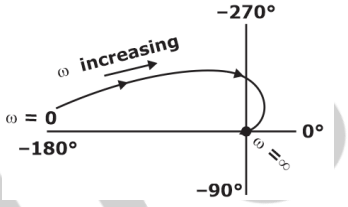
$$G(j\omega) = \frac{1}{j\omega^2(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{1}{\omega^2\angle -180°\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1\sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}\angle(-180° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -180°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -360°$




**Type: 2, Order: 5**

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$$

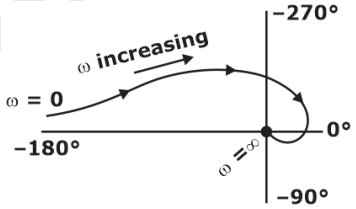
$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{1}{\omega^2\angle -180°\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1\sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2\sqrt{1+\omega^2T_3^2}\angle \tan^{-1}\omega T_3}$$

$$= \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}}\angle(-180° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3)$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -180°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -450° = 0\angle -90°$



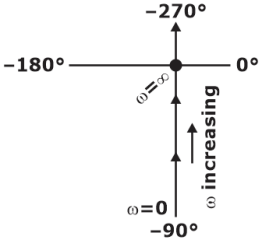

**Type: 1, Order : 1**

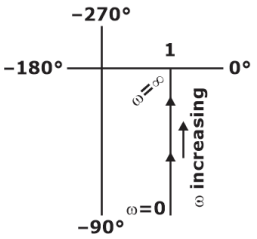
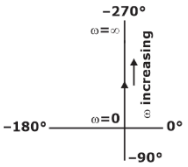
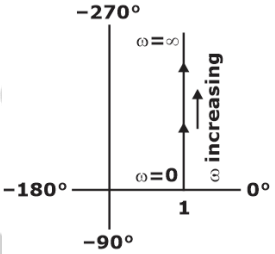
$$G(s) = \frac{1}{s}$$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega\angle 90°} = \frac{1}{\omega}\angle -90°$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90°$



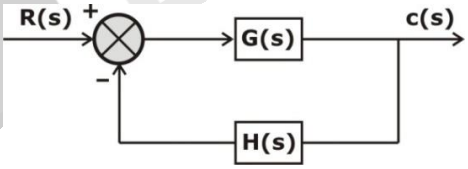
| | |
|---|---|
| $G(s) = \frac{1+sT}{sT}$<br>$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T} + 1 = \frac{1}{\omega T \angle 90^\circ} + 1 = \frac{1}{\omega T} \angle -90^\circ + 1$<br>As $\omega \to 0$, $G(j\omega) \to \infty \angle -90^\circ + 1$<br>As $\omega \to \infty$, $G(j\omega) \to 0 \angle -90^\circ + 1$ |  |
| G(s) = s<br>$G(j\omega) = j\omega = \omega \angle 90^\circ$<br>As $\omega \to 0$, $G(j\omega) \to 0 \angle 90^\circ$<br>As $\omega \to \infty$, $G(j\omega) \to \infty \angle 90^\circ$ |  |
| G(s) = 1 + sT<br>$G(j\omega) = 1 + j\omega T = 1 + \omega T \angle 90^\circ$<br>As $\omega \to 0$, $G(j\omega) \to 1 + 0 \angle 90^\circ$<br>As $\omega \to \infty$, $G(j\omega) \to 1 + \infty \angle 90^\circ$ |  |

## Nyquist Plot

It is the mapping of Nyquist contour into GH plane i.e. corresponding to the entire Nyquist contour, the Nyquist plot G(s)H(s) is plotted.

**Nyquist stability criterion (NSC):**

Consider a closed loop control system as shown in figure below:




The closed loop transfer function T(s) of the above system is given by

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

The poles of 1 + G(s) H(s) gives the same number of poles as of G(s)H(s) i.e. open-loop system and the zeroes of 1 + G(s)H(s) gives the same number of poles of T(s) i.e. closed loop system. On comparing the closed loop transfer function with

$$T(s) = \frac{(s-z_1)(s-z_2)(s-z_3)....}{(z-p_1)(s-p_2)(s-p_3)....}$$

$p_1$, $p_2$, $p_3$, ……are the poles of closed loop system.

$z_1$, $z_2$, $z_3$, ……..are the zeroes of the closed loop system.

By drawing the polar plot and completing closed loop contour from −∞ to 0 to +∞.

Nyquist stability criteria states that the number of encirclements about the critical point (-1+j0) should be equal to poles of open loop transfer function G(s)H(s), which are in the right half of s-plane.

Mathematically, N = P – Z

If Z = 0 then N = P

Where,

N is equals to the number of anticlockwise encirclement

P is the number of open loop poles lie in Right half of s plane &

Z is the number of closed loop poles lie in Right half of s plane

**NOTE:**

open loop system must be stable if P=0

close loop system must be stable if Z=0

## BODE PLOT

The step by step procedure for plotting the magnitude plot is given below:

**Step 1 :** Convert the transfer function into Bode form or time constant form. The Bode form of the transfer function is

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)\left(1+\frac{s^2}{\omega_n^2}+2\xi\frac{s}{\omega_n}\right)} \xrightarrow{s=j\omega} G(j\omega)$$

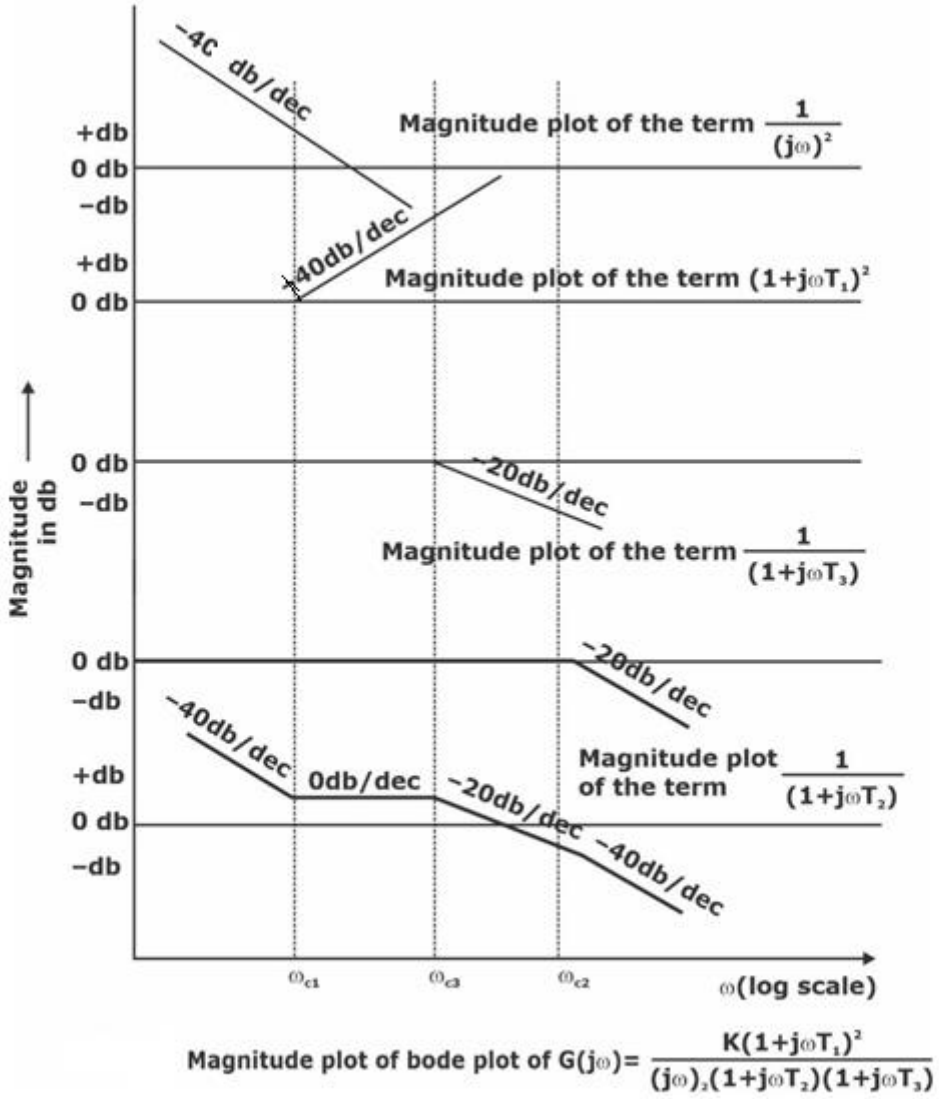
$$= \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)\left(1-\frac{\omega^2}{\omega_n^2}+j2\xi\frac{\omega}{\omega_n}\right)}$$

**Step 2 :** List the corner frequencies in the increasing order and prepare a table as shown below.

| Term | Corner frequency rad/sec | Slope db/dec | Change in slope db/dec |
|---|---|---|---|
| | | | |

In the above table enter K or $K/(j\omega)^n$ or $K(j\omega)^n$ as the first term and the other terms in the increasing order of comer frequencies. Then enter the comer frequency, slope contributed by each term and change in slope at every comer frequency.

**Step 3:** Choose an arbitrary frequency to, which is lesser than the lowest comer frequency. Calculate the db magnitude of K or $K/(j\omega)^n$ or $K(j\omega)^n$ at $\omega$, and at the lowest comer frequency.



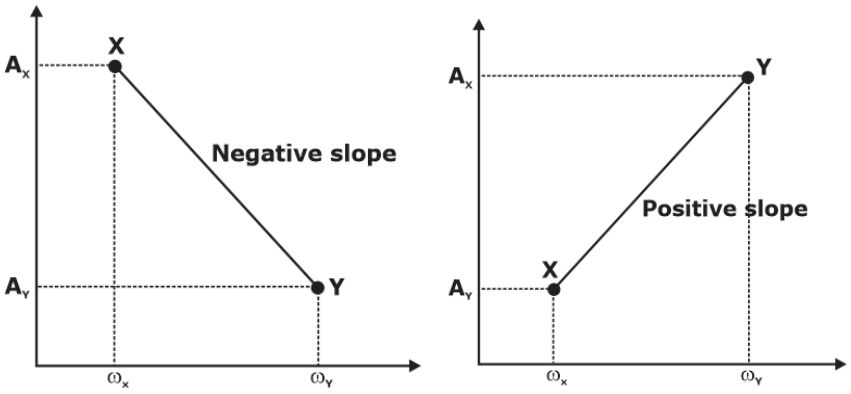

$$G(j\omega) = \frac{K(1 + j\omega T_1)^2}{(j\omega)^2 (1 + j\omega T_2)(1 + j\omega T_3)}$$

**Note:** 6dB/octave = 20 dB/decade

**Step 4 :** Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula,

Gain at $\omega_y$ = change in gain from $\omega_x$ to $\omega_y$ + Gain at $\omega_x$

$$= \left[\text{slope from } \omega_x \text{ to } \omega_y \times \log \frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$




**Step 5 :** Choose an arbitrary frequency $\omega_h$ . which is greater than the highest corner frequency. Calculate the gain at $\omega_h$ by using the formula in step 4.

**Step 6 :** In a semi log graph sheet mark the required range of frequency on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale) after choosing proper units.

**Step 7:** Mark all the points obtained in steps 3, 4, and 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

**Note:** The magnitude plot obtained above is an approximate plot. If an exact plot is needed, then appropriate corrections should be made at every corner frequency.

**PROCEDURE FOR PHASE PLOT OF BODE PLOT**

The phase plot is an exact plot and no approximations are made while drawing the phase plot. Hence the exact phase angles of $G(j\omega)$ are computed for various values of $\omega$ and tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually, the magnitude plot and phase plot are drawn in a single semi log - sheet on a common frequency scale.

Take another y-axis in the graph where the magnitude plot is drawn and in this y-axis mark the desired range of phase angles after choosing proper units. From the tabulated values of $\omega$ and phase angles, mark all the points on the graph. Join the points by a smooth curve.

**MINIMUM & NON- MINIMUM PHASE SYSTEM**

**1. Minimum Phase System:**

The transfer function with all poles and zeros in the left half of the s-plane are called minimum phase transfer function.

**2. Non-Minimum Phase System:**

The transfer function which has one or more zeros in the right half of the s- plane is known as non-minimum phase system.

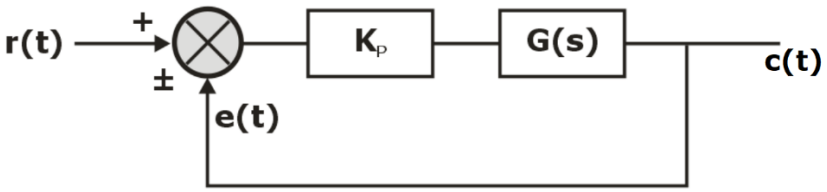
**3. All Pass System:**

The transfer function having a pole-zero pattern which is antisymmetric about the imaginary axis, i.e. for every pole in the left half plane, there is a zero in the mirror image position, is called as all-pass system.

# Controllers and Compensators

## 1. CONTROLLERS

### I. PROPORTIONAL CONTROLLER




**Proportional Controller**

Transfer function of a proportional controller is given as,

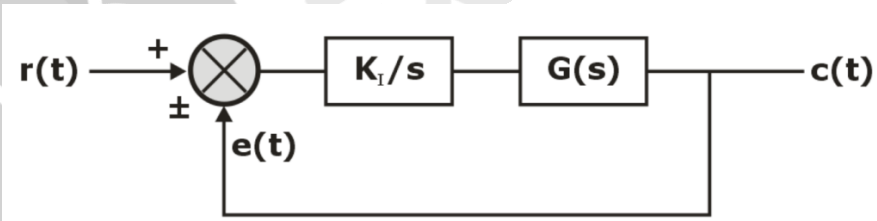
$G_C(s) = K_P$

**Effects of Proportional Controller:**

- Damping ratio decreases by factor $\sqrt{K_P}$
- Natural undamped frequency increases by factor $\sqrt{K_P}$
- Reduces the steady state error by increasing gain.

### II. INTEGRAL CONTROLLER

Integral control adds a pole at the origin for the open-loop system.




**Integral Controller**

Transfer function of Integral controller is given as,

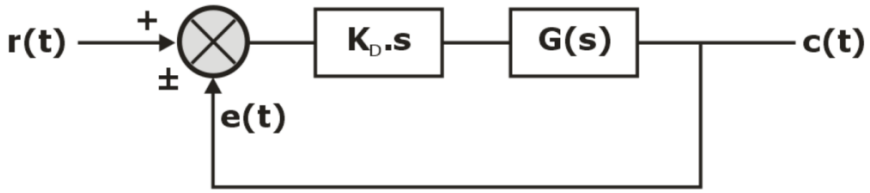
$$G_C(s) = \frac{K_I}{s}$$

**Effects of integral controllers:**

- It adds an open loop pole at origin.
- It increases the type and order of the system by 1.
- It Decrease the steady state error.
- It reduced the stability of the system.
- It improves the steady state performance.
- It acts like a low pass filter.

## III. DERIVATIVE CONTROLLER

Derivative controller adds a zero at the origin for the open-loop system.




**Derivative Controller**

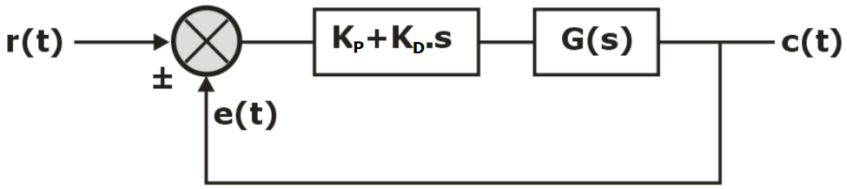
Transfer function of a derivative controller is given as,

$G_C(s) = K_D . s$

**Effects of derivative controller:**

- It adds an open loop zero at origin.
- It reduces the order and type of the system by 1.
- It is used to increase the stability of the system by adding zeros.
- Steady state error increases, as type of the system decreases.
- Transient response is improved.
- Differentiation makes the system sensitive to noise.

## IV. PROPORTIONAL + DERIVATIVE (PD) CONTROLLER




**Proportional Derivative Controller**

Transfer function of a PD controller is given as,

$G_C(s) = K_D . s + K_P$

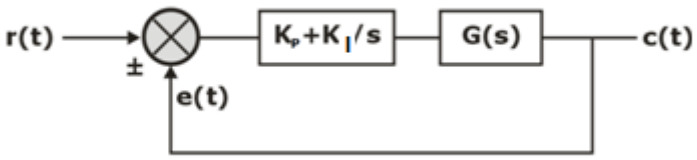
**Effects of PD Controller:**

- PD controller adds an open loop zero on negative real axis.
- It is used to increase the stability without affecting the steady state error.
- Damping ratio increases.
- Peak overshoot decreases.
- Transient response is improved.
- Bandwidth increases, so noise in the system increases.
- Settling time reduced.
- Rise time decreases.
- Delay time increases.
- Improves gain margin and phase margin.

## V. PROPORTIONAL + INTEGRAL (PI) CONTROLLER

Transfer function of PI controller is given as,

$$G_C(s) = K_P + \frac{K_I}{s}$$

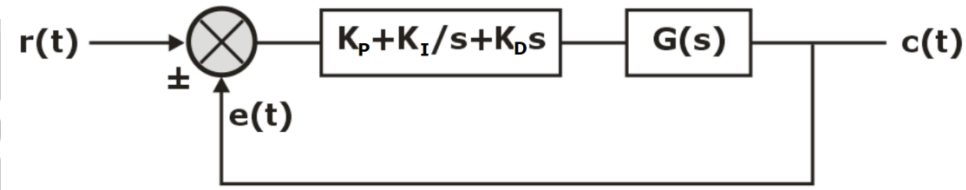



**PI controller**

**Effects of PI Controller:**

- It adds an open loop zero on negative real axis and open loop pole at origin.
- It increases type and order of the system by 1.
- As the type of the system is increases, therefore, steady state error is reduced, and hence steady state response is improved.
- As the order of the system increases, therefore, the relative stability decreases.
- Bandwidth reduced.
- Noise in the output is reduced and thus signal to noise ratio is increased.

## VI. PROPORTIONAL + INTEGRAL + DERIVATIVE (PID) CONTROLLER




**PID controller**

Transfer function of a PID controller is given as,

$$G_C(S) = K_P + K_I / s + K_D.s$$
$$= \frac{K_D.s^2 + K_P.s + K_I}{s}$$

**Effects of PID Controller:**

- It adds two open loop zeros in left half of s-plane and one open loop pole at origin.
- One zero compensates the pole and other zero will increase the stability.
- It improves both transient as well as steady state response.
- It is used to decrease the steady state error.
- Response is faster and highly accurate.

## CHARACTERISTICS OF P, I, AND D CONTROLLERS

A proportional controller ($K_P$) will have the effect of reducing the rise time but never eliminate the steady-state error.

An integral control ($K_I$) will have the effect of eliminating the steady-state error, but it may make the transient response worse.

A derivative control ($K_D$) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Effects of each of controllers $K_P$, $K_D$, and $K_I$ on a close-loop system are summarized in the table shown below.

| Closed Loop Response | Rise Time | Overshoot | Setting Time | Steady State Error |
|---|---|---|---|---|
| $K_P$ | Decrease | Increase | Small Change | Decrease |
| $K_I$ | Decrease | Increase | Increase | Eliminate |
| $K_D$ | Small Change | Decrease | Decrease | Small Change |

## 2. COMPENSATORS

There are two types of compensators.

1. Lead compensator

2. Lag compensator

## LEAD COMPENSATOR

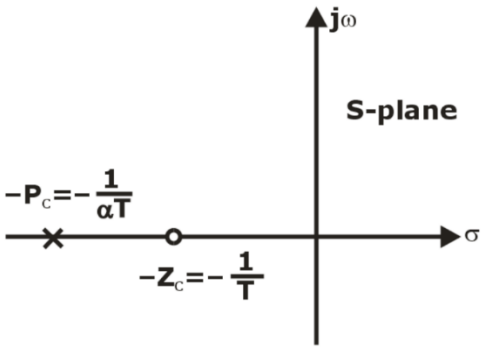
The lead compensator has the transfer function of the form:

$$G_c(s) = \frac{s + Z_c}{s + P_c} = \frac{s + \frac{1}{T}}{s + \frac{1}{aT}}$$

Where, $a = \frac{Z_c}{P_c} < 1$ or $Z_c < P_c$

And T > 0

Hence, we have the pole-zero plot as shown below:



## Pole-Zero Plot of Lead Compensator

### Maximum Phase Lead for Lead Compensator:

For lead compensator, the maximum phase angle is given by

$$\phi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right);\ \alpha < 1$$

### EFFECTS OF LEAD COMPENSATOR:

- The general effect is to add more damping to the closed loop system.
- Rise time and settling time are reduced.
- Phase margin improves.
- Improvement in relative stability.
- The bandwidth of the closed-loop system is increased; this corresponds to faster time response.
- Steady state error of the system is not affected.

### LAG COMPENSATOR

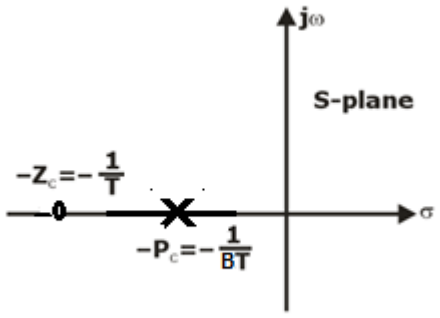
The lag compensator having a transfer function of the form:

$$G_c(s) = \frac{s+Z_c}{s+Z_p} = \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Where, $\beta = \frac{Z_c}{P_c} > 1$ or $Z_c > P_c$

and T > 0

Hence, we have the pole-zero plot as shown below:




## Pole-Zero Plot of Lag Compensator

### Maximum Phase Angle for Lag Compensator:

For lag compensator, the maximum phase angle is given by

$$\phi_m = \sin^{-1}\left(\frac{1-\beta}{1+\beta}\right);\ \beta > 1$$

**EFFECTS OF LAG COMPENSATOR:**

- Gain crossover frequency is decreased and thus the Bandwidth of the system is reduced.
- Magnitude of the forward path transfer function is attenuated near above the gain crossover frequency, thus improving the relative stability.
- The rise time and settling time of the system are usually longer.
- The system is more sensitive to parameter variations because the sensitivity function is greater than unity for all frequency (approximately greater the bandwidth).

# State Space Analysis

**Important Definitions:**

**1) State:** The state of a dynamic system is defined as a minimal set of variables such that the knowledge of input for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

**2) State Variables:** The variables involved in determining the state of a dynamic system X(t), are called the state variables. $X_1(t)$, $X_2(t)$....$X_n(t)$ are nothing but the state variables. These are normally the energy storing elements contained in the system.

**3) State Vector:** The 'n' state variables necessary to describe the complete behaviour of the system can be considered as 'n' components of a vector X(t) called the state vector at time 't'. The state vector X(t) is the vector sum of all the state variables.

**4) State Space:** The space whose co-ordinate axes are nothing but the 'n' state variables with time as the implicit variable is called the state space.

**5) State Trajectory:** It is the locus of the tips of the state vectors, with time as the implicit variable.

**State Equation:**

$$\dot{X}(t) = AX(t) + BU(t)$$

where , X(t) = State vector matrix of order n × 1

U(t) = Input vector matrix of order m × 1

A = System matrix or Evolution matrix of order n × n

B = Input matrix or control matrix of order n × m

**Output equations:**

$$Y(t) = C\,X(t) + D\,U(t)$$

Where, Y(t) = Output vector matrix of order p × 1

C = Output matrix or observation matrix of order p × n

D = Direct transmission matrix of order p × m

**Transfer Function Matrix:**

$$C[sI - A]^{-1}B + D = H(s)$$

**Transfer Matrix Description:**

Y(s) = H(s)U(s)

**State Space Methods:**

1. General State Equation Solution:

$$x(t) = e^{At-t_o}x(t_o) + \int_{t_o}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$X(s) = \underbrace{[sI - A]^{-1}X[0]}_{\text{zero input response}} + \underbrace{[sI - A]^{-1}BU(s)}_{\text{zero state response}}$$

$$x[n] = A^n x[0] + \sum_{m=0}^{n-1} A^{n-1-m}Bu[n]$$

2. General Output Equation Solution:

$$y(t) = Ce^{At-t_o}x(t_o) + C\int_{t_o}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$y[n] = CA^n x[0] + \sum_{m=0}^{n-1} CA^{n-1-m}Bu[n] + Du[n]$$

$$Y(s) = \underbrace{C\left[sI - A\right]^{-1} X\left[0\right]}_{\text{zero input response}} + \underbrace{\left[C\left[sI - A\right]^{-1} B + D\right]U(s)}_{\text{zero state response}}$$

**STATE-TRANSITION MATRIX**

$$e^{At} = \phi(t) = L^{-1}\left[\left[sI - A\right]^{-1}\right]$$

**Properties of the State-Transition Matrix**

The state-transition matrix STM possesses the following properties:

1. $\phi(0) = I$ (the identity matrix)
2. $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_2 - t_1)\,\phi(t_1 - t_0) = \phi(t_2 - t_0)$ for any $t_0$, $t_1$, $t_2$
4. $(t)^k = \phi(kt)$ for k = positive integer
5. $\dot{\phi}(t) = A\phi(t)$

**Controllability**

$Q_c = [B : AB : A^2B : \ldots\ldots\ldots\ldots A^{n-1}B]$

Where, $Q_c$ = controllability test matrix (n x nm)

Condition for state controllability,

$|Q_c| \neq 0$ (Matrix should be non-singular)

**Observability**

$Q_o = [C^T : A^TC^T : (A^T)^2C^T : \ldots\ldots\ldots\ldots (A^T)^{n-1}C^T]$

Where, $Q_o$ = observability test matrix (n x np)

Condition for state observability,

$|Q_0| \neq 0$ (Matrix should be non-singular)

**NOTE:**

If A is a diagonal matrix then

(i) System is observable if matrix C has any non zero element.
(ii) System is controllable if matrix B has any non zero element.

BYJU'S EXAM PREP

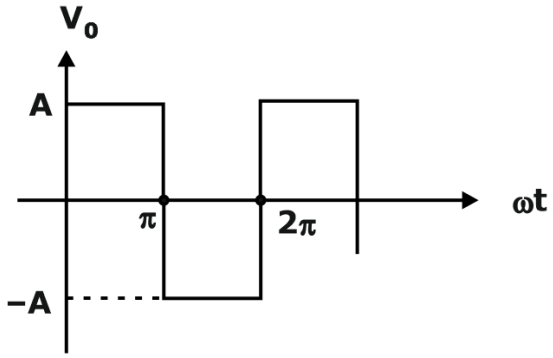
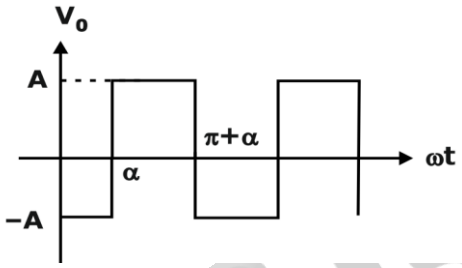
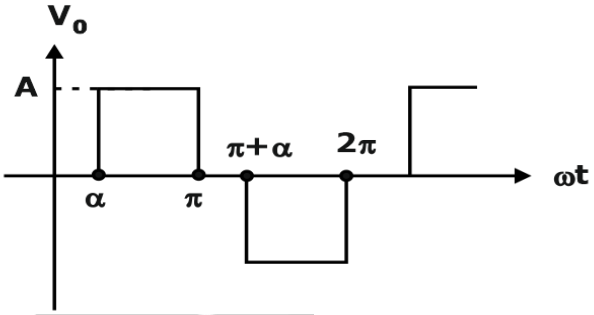
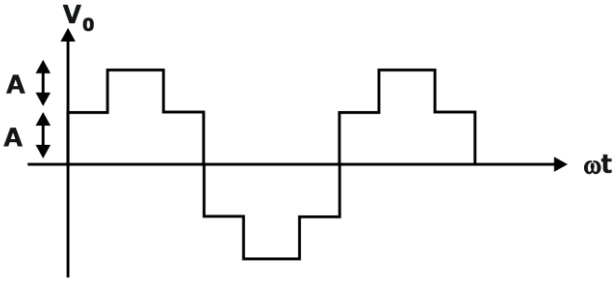
# Electrical Engineering

## Basic of Power Electronics

SHORT NOTES

# 1. Basics of power electronics

**Important Fourier series representations:**

| S. No | Signal | Fourier series expansion |
|---|---|---|
| 1. |  | $V_0 = \sum_{n=1,3,5} \frac{4A}{n\pi} \text{Sin} n\omega t$ |
| 2. |  | $V_0 = \sum_{n=1,3,5} \frac{4A}{n\pi} \text{Sin}(n\omega t - n\alpha)$ |
| 3. |  | $V_0 = \sum_{n=1,3,5} \frac{4A}{n\pi} \text{Cos}\left(\frac{n\alpha}{2}\right) \text{Sin}(n\omega t - \frac{n\alpha}{2})$ |
| 4. |  | $V_0 = \sum_{n=1,5,7} \frac{6A}{n\pi} \text{Sin} n\omega t$ :<br>Each pulse width of $60^0$ Duration |

- For output of half wave uncontrolled rectifier Fourier series expression is

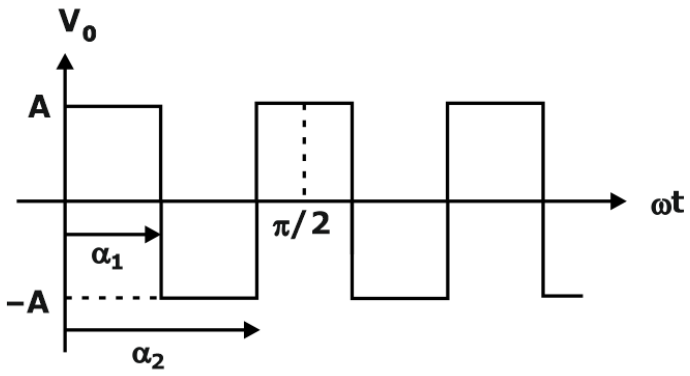
$$V_0 = \frac{A}{\pi} + \frac{A}{2}\text{Sin}\,\omega t + \sum_{n=2,4,6} \frac{2A}{\pi(1-n^2)}\cos n\omega t$$ ; Where A= Amplitude of signal

- For output of Full wave uncontrolled rectifier Fourier series expression is

$$V_0 = \frac{2A}{\pi} + (\text{fundamental} = 0) + 2\sum_{n=2,4,6} \frac{2A}{\pi(1-n^2)}\cos n\omega t$$

- Two switching quarter per cycle (topic of inverters):

Fourier series expansion is $V_0 = \frac{4A}{n\pi}\left(1 - 2\cos n\alpha_1 + 2\cos n\alpha_2\right)$



## 2. Power semiconductor devices

| S.No | Device | Features |
|---|---|---|
| 1 | Diode | • Uncontrolled device<br>• Unipolar and unidirectional |
| 2 | BJT/MOSFET/IGBT | • Fully controlled<br>• Unipolar and unidirectional (Without body diodes) |
| 3 | SCR and GTO | • SCR-Semi controlled<br>• GTO- Fully controlled<br>• Both are Bipolar and unidirectionaL |
| 4 | TRIAC | • Semi controlled device<br>• Bipolar and Bidirectional |

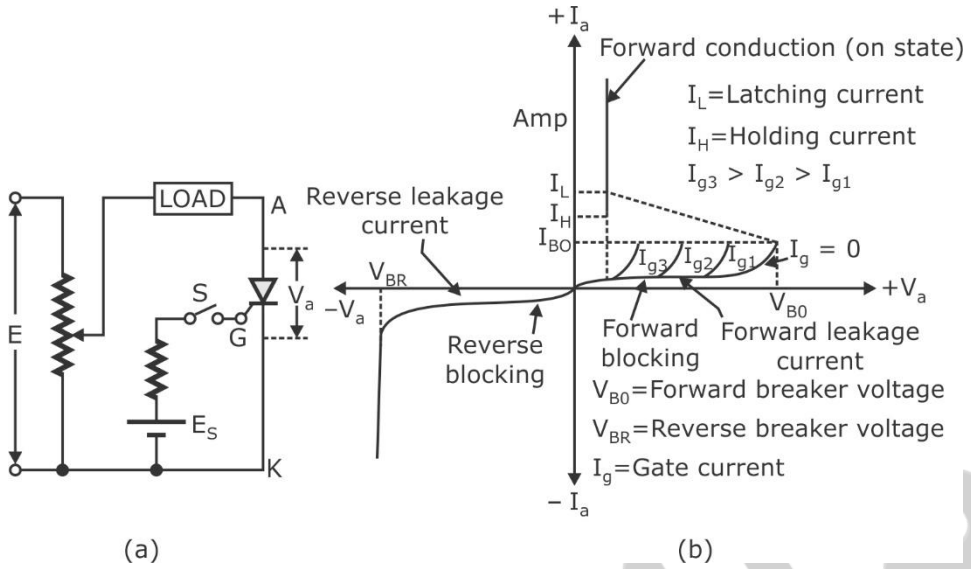
- Conduction losses in BJT is less than MOSFET.
- Switching time in MOSFET is less than BJT.
- Majority carrier devices: MOSFET and Schottky diode are having positive temperature coefficient property.
- Minority carrier devices: SCR, BJT, GTO, IGBT, Power diode and having Negative Temperature Coefficient property.
- IGBT was developed in 1988 by combining the advantage of both BJT and MOSFET. IGBT possesses high input impedance like a MOSFET and has low on-state voltage as in a BJT. It does not suffer from secondary breakdown problems. The turn-on speed can be controlled by the rate of change of gate-source voltage.
- The turn-off time of IGBT consists of three intervals: (i) delay time, (ii) initial fall time (iii) final fall time.
- SIT is a high-power high-frequency solid-state semiconductor device. It is basically $n^+ n^- n$ device with a buried grid-like $p^+$ gate structure
- The SIT has been used in audio, VHF/UHF and microwave amplifiers. The reliability, noise, switching speed and radiation hardness of SIT are claimed to be superior to MOSFET. It has lower gate-to-source capacitance and resistance.
- SCR is a four-layer, three junctions, p-n-p-n switching device. It has three terminals namely anode, cathode, and gate. It is called SCR because silicon is used for construction and its operation is similar to a rectifier i.e., very low resistance in forward conduction and very high resistance in a reverse direction.
- SCR is a unidirectional device as the current can be low only from anode to cathode and it is a bilateral switch which means it can block the voltage of either direction. SCR does not allow conduction in forward direction until its gate terminal gets a proper triggering signal.
- SCR has three modes of operations:
  i. Forward blocking mode
  ii. Forward conduction mode
  iii. Reverse blocking mode

- SCR can be turned-on by following methods:
  i. Forward voltage triggering
  ii. $\frac{dv}{dt}$ triggering
  iii. gate triggering
  iv. light triggering
- SCR turn-off means that it has changed from on to off state. SCR turn-off contains two steps:

(a) Reducing the anode current below holding current
(b) Removal of stored charges from the semiconductor layer

- Circuit turn-off time ($t_c$) is defined as the time between the instant anode current becomes zero and the instant reverse voltage due to the practical circuit reaches zero.
- For reliable turn-off $t_c$ should be greater than $t_q$, otherwise the device may turn on at an undesired instant, a process called commutation failure.
- For reliable and satisfactory operation of SCR, it must be protected against all abnormal conditions. These conditions are over voltages, over currents, high $\frac{di}{dt}$, false triggering due to high value of $\frac{dv}{dt}$ and spurious signal across gate-cathode terminal may lead to unwanted turn-on.
- By using a small inductor called a limiting inductor in series with the anode circuit can be used for $\frac{di}{dt}$ protection.
- If the rate of rise of the suddenly applied voltage across the thyristor is high the device may get turned on. Such phenomena of turning on a thyristor called $\frac{dv}{dt}$ turn-on, it must be avoided as it leads to false operation. False turn-on of an SCR by large change of voltage even without application of gate signal can be prevented by using a snubber circuit in parallel with the device.
- To suppress these over voltages, voltage clamping device is used. a voltage clamping device is a non-linear resistor connected across SCR. Voltage clamping device has falling resistance characteristic with increasing voltage. Under normal conditions the device has a high resistance and draws only a small leakage current. When a voltage surge appears, the voltage clamping device operates in lower assistance region and produces a short circuit across SCR.
- To protect SCR from overcurrent we can use two devices namely fast acting current limiting fuse (FACLF), circuit Breakers.
- During conduction power loss occurs in SCR due to
  i) Forward conduction loss
  ii) gate triggering loss
  iii) Switching loss
  iv) loss due to leakage current

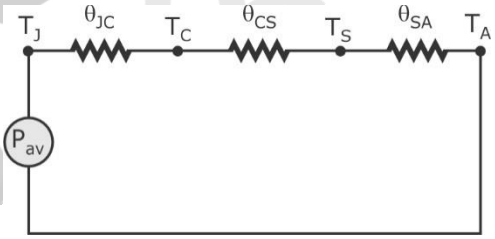
1.Static V-I characteristics of SCR:




- Latching Current: Minimum current required for conduction even after the gate pulse is removed.
- Holding Current: Minimum Current below which SCR is turned off
- Usually Latching current is 1.5 to 3 times of Holding current

2.Thermal equivalent circuit:

$\theta_{JC}$ is the thermal resistance between the junction and case
$\theta_{CS}$ is the thermal resistance between the case and sink
$\theta_{SA}$ is the thermal resistance between the sink and ambiance




$$P_{avg} = \frac{T_J - T_C}{\theta_{JC}} = \frac{T_C - T_S}{\theta_{CS}} = \frac{T_S - T_A}{\theta_{SA}}$$

Rating of thyristor $\propto \sqrt{P_{avg}}$

3.Charge stored in depletion region:

Let $Q_R$ be the charge stored in depletion region of power diode.

$$Q_R = \frac{1}{2}.I_{RM}.t_{rr}$$

$$I_{RM} = \frac{2Q_R}{t_{rr}} = t_a.\frac{di}{dt}$$

If $t_a \approx t_{rr}$, $t_{rr} = \sqrt{\frac{2Q_R}{di/dt}}$

$I_{RM} = t_{rr} \cdot \frac{di}{dt} = \sqrt{2Q_R \left(\frac{di}{dt}\right)}$

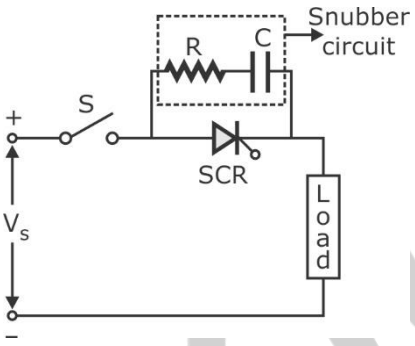
$t_{rr} \propto \sqrt{Q_R}$

$Q_R \propto I_f$

$t_{rr} \propto \sqrt{I_f}$

$I_{RM} \propto \sqrt{I_f}$

4.Design of Snubber circuit:




For Inductor $(L)$:

$\left(\frac{di}{dt}\right)_{max} = \frac{V_s}{L}$

$L = \frac{V_s}{(di/dt)_{max}}$

For resistor $(R)$:

$\left(\frac{dv_s}{dt}\right)_{max} = R\left(\frac{di}{dt}\right)_{max}$

or $R = \frac{L}{V_s}\left(\frac{dv_s}{dt}\right)_{max}$

For Capacitor $(C)$:

$C = \left(\frac{2\xi}{R}\right)^2 L$ where $0.5 < \xi < 1$

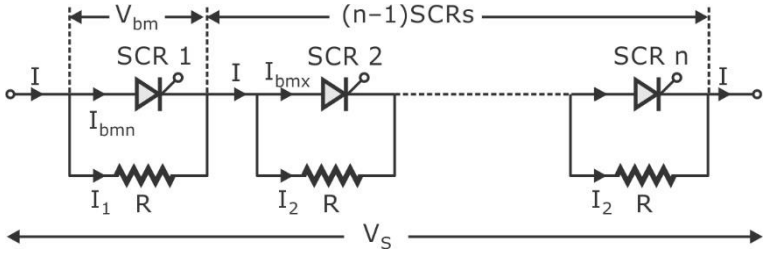
5.Series and parallel operation of SCR:

- String efficiency $= \frac{\text{Actual voltage/current rating of string.}}{n \times \text{individual voltage/current rating of SCR}}$

  Where n is the number of SCR in string.

  Derating factor, DRF=1- string efficiency.

- Series operation:

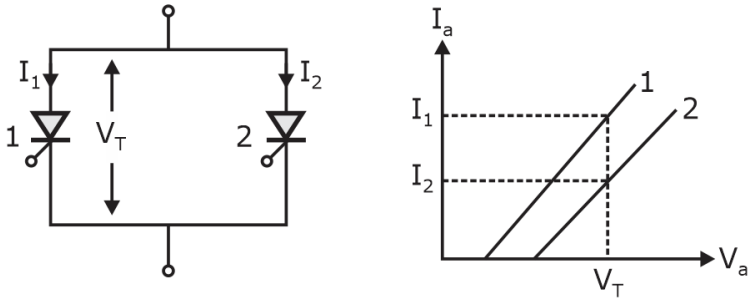



Static equalizing resistance $R = \frac{nV_{bm} - V_s}{(n-1)\Delta I_b}\ \Omega$

Dynamic equalizing capacitance $C = \frac{(n-1)\Delta Q}{nV_{bm} - V_s}\ F$

- Parallel operation.

  When current required by the load is more than the rated current of a single thyristor, SCRs are connected in parallel.




(a) (b)

String efficiency $= \frac{I_1 + I_2}{2I_1} = \frac{1}{2}\left(1 + \frac{I_2}{I_1}\right)$

6.Ratings of Thyristors:

1)$I_{Trms}$ Rating: The actual Thyristor rms in a converter must always be less than thyristor RMS ratings.

$I_{Trms}$ value in a converter < $I_{frms}$ rating.

2)$I_{Tavg}$ Rating: (average on-state current ratings)

$$(I_{Tavg}) \text{ rating} = \frac{(I_T)_{rms} \text{ Rating}}{\text{Form Factor of thyristor current waveform}}$$

Average rating of a thyristor depends on:

- Conduction angle of thyristor increases which decrease the form factor and then increase the average thyristor rating.
- Type of load: Smoothness of thyristor current waveform increase the FF decreases and therefore $(I_{Tavg})_{Rating}$ increases.
- Type of converter: because FF of thyristor waveform depends on average value of converter.

3)$I^2t$ Rating of thyristor: specified to select a proper fuse for overcurrent protection.

$I^2t$ current Rating of thyristor > $I^2t$ current Rating of Fuse.

4)Surge current rating of thyristor:

General values

$(I_T)_{rms}$ = 35A

$(I_S)_{rms}$ = 2000A for one cycle and 3000A for 2MW

N-cycle surge current rating: $(I_m)$: It is the surge current that the SCR can withstand for n-cycles.

$\boxed{(I_{Sn})^2 \left(\frac{nT}{2}\right) = I^2t \text{ rating of thy}}$ from the equation, we can find the value of '$I_{sn}$'

One-cycle surge current rating (Is): It is the surge current that the SCR can withstand for a cycle. $I_{s1}^2 = (I_{sn})^2$ $\boxed{I_{S_1} = \sqrt{n}\, I_{sn}}$

Sub-cycle surge current rating: It is the surge current that the SCR can withstand for 1/nth period of a cycle. $(I_{s/n})^2 = (I_{sn})^2$ $\boxed{\frac{I_s}{n} = \sqrt{n}\, I_{s1}}$

## 3. RECTIFIERS

- For n-pulse converter:

  Source current has $nk \pm 1$ Harmonics k=1,2,3…….

  Output voltage has nk Harmonics.

1.Single Phase Half Wave controlled rectifier:

**R-load:**

Average output voltage $V_{o,avg} = \frac{V_m}{2\pi}(1+\cos\alpha)$

RMS output voltage $V_{o,rms} = \sqrt{\frac{V_m^2}{4\pi}\left((\pi-\alpha)+\frac{\sin 2\alpha}{2}\right)}$

**RL-Load:**

Average output voltage $V_{o,avg} = \frac{V_m}{2\pi}(\cos\alpha - \cos\beta)$

RMS output voltage $V_{o,avg} = \frac{V_m}{2\sqrt{\pi}}\left(\sqrt{(\beta-\alpha)+\frac{1}{2}(\sin 2\alpha - \sin 2\beta)}\right)$

Circuit Turnoff time: $t_c = \frac{2\pi-\beta}{\omega}$

**For a constant output current refer below table:**

| | 1-$\phi$ Full conv. | 3 $\phi$ Full conv. | 1 $\phi$ Semi conv. | 3 $\phi$ semi conv. |
|---|---|---|---|---|
| Output voltage | $\frac{2V_m}{\pi}\cos\alpha$ | $\frac{3V_{m,line}}{\pi}\cos\alpha$ | $\frac{V_m}{\pi}(1+\cos\alpha)$ | $\frac{3V_{m,line}}{2\pi}(1+\cos\alpha)$ |
| Fundamental source current RMS ( $I_{s1}$ ) | $\frac{2\sqrt{2}}{\pi}I_0$ | $\frac{\sqrt{6}}{\pi}I_0$ | $\frac{2\sqrt{2}}{\pi}I_0\cos\left(\frac{\alpha}{2}\right)$ | $\frac{\sqrt{6}}{\pi}I_0\cos\left(\frac{\alpha}{2}\right)$ |
| Source current RMS ( $I_s$ ) | $I_0$ | $\sqrt{\frac{2}{3}}I_0$ | $I_0\sqrt{\frac{\pi-\alpha}{\pi}}$ | $\alpha \le 60^0 \rightarrow I_0\sqrt{\frac{2}{3}}$<br>$\alpha > 60^0 \rightarrow I_0\sqrt{\frac{\pi-\alpha}{\pi}}$ |
| Displacement power factor (DPF) | $\cos\alpha$ | $\cos\alpha$ | $\cos\left(\frac{\alpha}{2}\right)$ | $\cos\left(\frac{\alpha}{2}\right)$ |

➢

- Distortion factor (DF) = $\frac{I_{s1}}{I_s}$
- Input power Factor= DF*DPF
- Total Harmonic distortion = $\sqrt{\left(\frac{1}{DF}\right)^2 - 1}$

**For R-Load refer below table:**

| | 3 ϕ Half wave rectifier | 3 ϕ full wave rectifier |
|---|---|---|
| Continuous | $\alpha < 30^0$<br>$\frac{3V_{m,line}}{2\pi}\text{Cos}\,\alpha$<br>**Hint:**<br>Integrate from 30+α to 150+ α and Time period T=120$^0$ and function take in phase, you will get above formula<br>Like this<br>$\frac{1}{\left(\frac{2\pi}{3}\right)}\int_{30+\alpha}^{150+\alpha} V_{m,phase}\,\text{Sin}\,\omega t\,dwt$ | $\alpha < 60^0$<br>$\frac{3V_{m,line}}{\pi}\cos\alpha$<br>**Hint:**<br>Integrate from 60+ α to 120+ α and Time period T=60$^0$ and function take in line, you will get above formula |
| Discontinuous | $\alpha \geq 30^0$<br>$V_0 = \frac{3V_{m,phase}}{2\pi}\left(1+\cos\left(\alpha+\frac{\pi}{6}\right)\right)$<br>**Hint:**<br>Integrate from 30+ α to 180$^0$ and Time period T=120$^0$ and function take in phase, you will get above formula | $\alpha \geq 60^0$<br>$V_0 = \frac{3V_{m,line}}{\pi}\left(1+\cos\left(\alpha+\frac{\pi}{3}\right)\right)$<br>**Hint:**<br>Integrate from 60+ α to 180$^0$ and Time period T=60$^0$ and function take in line, you will get above formula |

**Effect of Source Inductance:**

- **1**ϕ Half wave:

$$V_0 = \frac{V_m}{2\pi}(1+\cos\alpha) - fL_sI_0$$

$$I_0 = \frac{V_m}{\omega L_S}(\cos\alpha - \cos(\alpha+\mu))$$

- **1** ϕ Full wave:

$$V_0 = \frac{2V_m}{\pi}\cos\alpha - 4fL_sI_0$$

$$I_0 = \frac{V_m}{2\omega L_S}(\cos\alpha - \cos(\alpha+\mu))$$

$$\text{Regulation} = \frac{\cos\alpha - \cos(\alpha+\mu)}{2\cos\alpha}$$

Displacement power factor: $\cos\left(\alpha+\frac{\mu}{2}\right)$

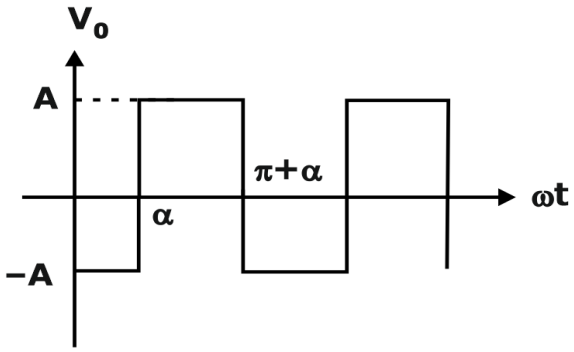
- $3\phi$ Full Wave:

$$V_0=\frac{3V_{m,line}}{\pi}\cos\alpha-6fL_sI_0$$

$$I_0=\frac{V_{m,line}}{2\omega L_S}\left(\cos\alpha-\cos(\alpha+\mu)\right)$$

**Single Phase Full converter:**

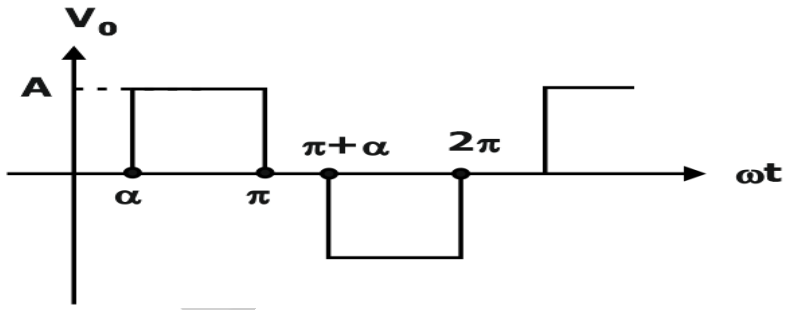
Source current waveform:




Fundamental source current is $i_{s1}=\frac{2\sqrt{2}}{\pi}I_0$

**Single phase half controlled or Semi converter:**

Source current waveform:




- In this there are two configurations:

  Symmetrical configuration: On one leg one thyristor and one diode

  Unsymmetrical configuration: on one leg two thyristors or two diodes

  $\gamma$ represents conduction in below table

  D- Diode, T-Thyristor, $\alpha$-Firing angle

| Symmetrical configuration | Unsymmetrical configuration | Full converter with Freewheeling diode |
|---|---|---|
| $\gamma_T=\pi$ | $\gamma_T=\pi-\alpha$ | $\gamma_T=\pi-\alpha$ |

| | | |
|---|---|---|
| $\gamma_D = \pi$ | $\gamma_D = \pi + \alpha$ | $\gamma_D = 2\alpha$ |

**3-Phase**:

**3Phase half wave-controlled rectifiers:**

- Take phase voltage reference in the integration function for the below mentioned limits for calculations
- For R-Load $\alpha < 30^0$

  $\alpha < 30^0$ : Continuous conduction: 30+ $\alpha$ to 150+ $\alpha$

  $\alpha \geq 30^0$ : Discontinuous conduction: 30+ $\alpha$ to $180^0$
- For current stiff load:

  Without Freewheeling diode: 30+ $\alpha$ to 150+ $\alpha$

  With Freewheeling diode: $\alpha < 30^0$ :30+ $\alpha$ to 150+ $\alpha$

  $\alpha \geq 30^0$ : 30+ $\alpha$ to $180^0$
- Mentioned limits are useful while calculating output voltage average or RMS values for those particular conditions. Use phase as reference while doing calculations of average and RMS

**3Phase full wave-controlled Rectifiers:**

- Take Line voltage reference in the integration function for the below mentioned limits for calculations
- Limits are 60+ $\alpha$ to 120+ $\alpha$ for calculating output voltage average or RMS value
- Circuit Turnoff time:

$$\alpha \leq 60^0,\ t_c = \frac{240^0 - \alpha}{\omega}$$

$$\alpha > 60^0,\ t_c = \frac{180^0 - \alpha}{\omega}$$

**3Phase semi converter:**

$\alpha < 60^0$ it is 6 Pulse converter

$\alpha \geq 60^0$ it is 3 Pulse converter

- Freewheeling Action Duration:

  $\alpha < 60^0$, Duration = zero (No freewheeling action)

  $\alpha \geq 60^0$, Duration= $3\left(\alpha - \frac{\pi}{3}\right)$
- Limits for calculating output voltage average or RMS values (Line voltages are reference)

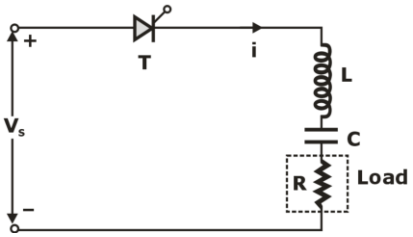
  $\alpha < 60^0$ : $60^0 + \alpha$ to $120^0 \rightarrow V_{AB}$ reference

  : $120^0$ to $180^0 + \alpha \rightarrow V_{AC}$ reference

  $\alpha \geq 60^0$ : $60^0 + \alpha$ to $240^0 \rightarrow V_{AC}$ reference

## 4.Commutation Techniques

1) **Class A Commutation (Load Commutation/self-commutation)**



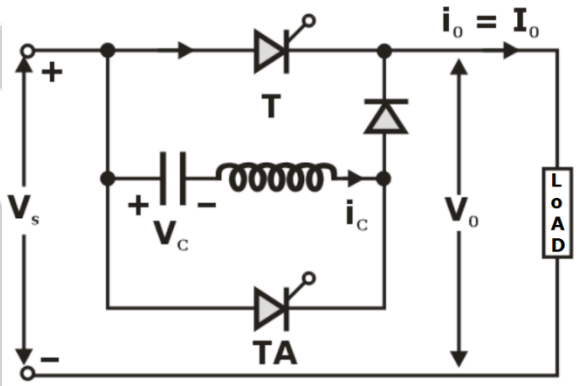

- For successful Load commutation, circuit must be under damped
- For under damped $I = I_p e^{-\alpha t} \operatorname{Sin} \omega_r t$

$$I_p = \frac{V_s}{\omega_r L}, \alpha = \text{Dampingfactor} = \frac{R}{2L}, \ \omega_r^2 = \omega_0^2 - \alpha^2$$

$\omega_r$ is ringing frequency, $\omega_0 = \text{Naturalfrequency} = \frac{1}{\sqrt{LC}}$

- Conduction time of thyristor, $t_c = \frac{\pi}{\omega_r}$

2) **Class B Commutation/Current Commutation/Resonant pulse commutation:**




- Voltage across capacitor $V_c = V_s \cos \omega_0 t$
- Circuit turn-off time for the main thyristor $(T_1)$; $tc = C\frac{V_{ab}}{I_0}$

$$V_{ab} = V_s \cos \omega_0 (t_3 - t_2)$$

Where $t_3$ = time when the main thyristor is turned off

$t_2$= time when auxiliary thyristor is turned off

$$\omega_0 (t_3 - t_2) = \sin^{-1}\left(\frac{I_0}{I_p}\right)$$

- Main thyristor peak current $=I_0$
- Auxiliary Thyristor peak current= $V_s\sqrt{\frac{C}{L}}$
- Conduction time of auxiliary thyristor= $\pi\sqrt{LC}$

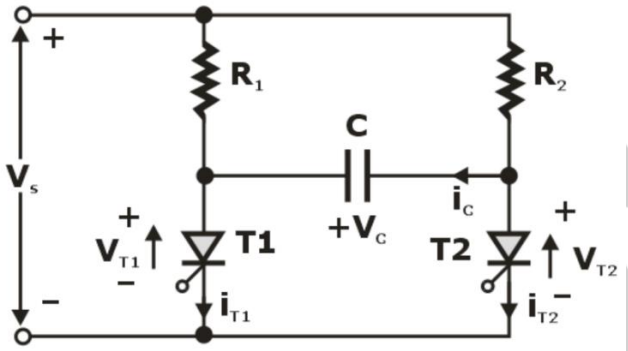
- Conduction time of main thyristor $= \pi\sqrt{LC} + \sqrt{LC}\sin^{-1}\left(\frac{I_0}{I_p}\right)$

  Minimum Conduction time of main thyristor= $\pi\sqrt{LC}$

  Maximum Conduction time of main thyristor = $\frac{3}{2}\pi\sqrt{LC}$

- Time for which capacitor current exists = $\frac{C}{I_0}(V_{ab} + V_s)$

3) **Class C Commutation (Impulse/Complementary commutation):**




When $T_1$ is turned on at t=0

- The charging current $I_s = \frac{V_s}{R_z}.e^{-t/R_2 C}$
- Voltage across capacitor

  $V_c(t) = V_s(1 - e^{-t/R_2 C})$

  When $T_1$ is to be turned-off, $T_2$ is turned-on at $T_1$
- The charging current $I_c(t) = -\frac{2V_S}{R_1}.e^{-t/R_1 C}$
- The Voltage across capacitor

  $V_c(t) = V_s[2e^{-t/R_1 C} - 1]$
- Maximum current though thyristor $T_1$

  $I_{T_1(max)} = V_s\left[\frac{1}{R_1} + \frac{2}{R_2}\right]$
- Maximum current though thyristor $T_2$,

  $I_{T_2(max)} = V_s\left[\frac{2}{R_1} + \frac{1}{R_2}\right]$

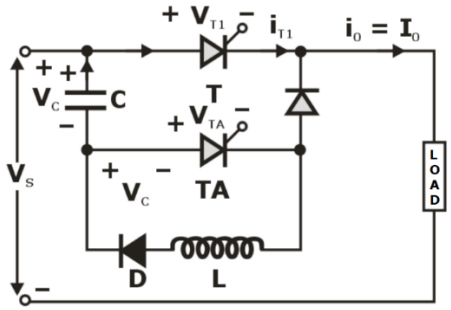
  Circuit turn-off time $t_{c_1}$ for thyristor $T_1$

  $t_{c_1} = R_1 C\ln(2)$

  Circuit turn-off time $t_{c_2}$ for thyristor $T_2$

  $t_{c_2} = R_2 C\ln(2)$

4) **Class D Commutation (Voltage commutation):**




- Maximum thyristor current Peak= $I_0 + V_s\sqrt{\frac{C}{L}}$
- Auxiliary Thyristor peak current= $I_0$
- Capacitor peak current = $V_s\sqrt{\frac{C}{L}}$
- Circuit turn-off time for main thyristor $T_1$ is $t_c = C\frac{V_s}{I_0}$

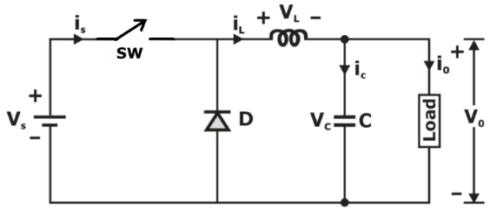
For R-load: $t_c$= RC ln2

- Circuit turn-off time for main thyristor (TA)

$$t_{c_1} = \frac{\pi}{2\omega_0}$$

## 5. DC-DC Converters

**Buck Converter:**




In Buck regulator, the average output voltage $V_0$ is less than the input voltage $V_S$.

$$\Delta I = \frac{(V_S - V_0)T_{ON}}{L}$$

$$\Delta I = \frac{V_0 T_{OFF}}{L}$$

$$V_0 = V_S\frac{T_{ON}}{T} = V_S\alpha$$

Where $\Delta I = I_2 - I_1$ is the peak-to-peak current ripple of the inductor L.

The peak-to-peak ripple current is $\Delta I = \frac{V_S\alpha(1-\alpha)}{fL}$

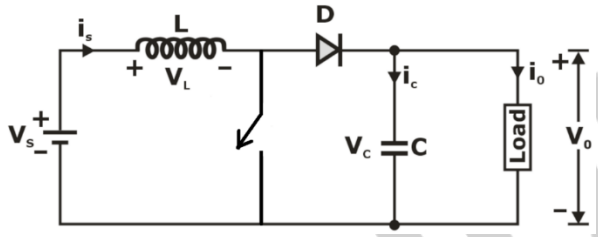
The peak to ripple voltage of the capacitor is $\Delta V_C = \frac{V_S \alpha (1-\alpha)}{8LCf^2}$

**Condition for continuous inductor current and capacitor voltage:**

If $I_L$ is average inductor current, the inductor ripple current $\Delta I = 2I_L$, which gives the critical value of the inductor $L_C$ as $L_C = L = \frac{(1-\alpha)R}{2f}$

If $V_C$ is the average capacitor voltage, the capacitor ripple voltage $\Delta V_C = 2V_0$, which gives the critical value of capacitor $C_C$ as $C_C = C = \frac{1-\alpha}{16Lf^2}$

**Boost Converter:**




$$\Delta I = \frac{V_S T_{ON}}{L} = \frac{(V_0 - V_S) T_{OFF}}{L}$$

where $\Delta I = I_2 - I_1$ is peak to peak ripple current of the inductor L.

The average output voltage,

$$V_0 = V_S \frac{T}{T_{OFF}} = \left(\frac{1}{1-\alpha}\right) V_S$$

The peak to peak current ripple is, $\Delta I = \frac{V_S \alpha}{fL}$

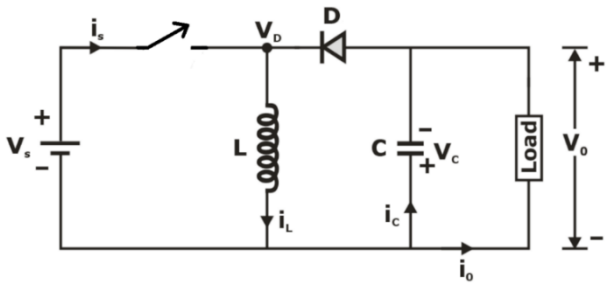
The peak to peak ripple voltage of capacitor, $\Delta V_C = \frac{I_0 \alpha}{fC}$

**Condition of continuous inductor current and capacitor voltage:**

If $V_C$ is the average capacitor voltage, the capacitor ripple voltage $\Delta V_C = 2V_0$, which gives the critical value of the capacitor $C_C$ as $C_C = \frac{\alpha}{2fR}$

If $I_L$ is average inductor current, the inductor ripple current $\Delta I = 2I_L$, which gives the critical value of the inductor $L_C$ as $L_C = L = \frac{\alpha(1-\alpha)^2 R}{2f}$

**Buck Boost Converter:**



$$\Delta I = \frac{V_S T_{ON}}{L} = \frac{-V_0 T_{OFF}}{L}$$

where $\Delta I = I_2 - I_1$ is the peak to peak ripple current of inductor L.

The average output voltage is, $V_0 = -\frac{V_S \alpha}{1-\alpha}$

The peak to peak current ripple is, $\Delta I = \frac{V_S \alpha}{fL}$

peak to peak ripple voltage of the capacitor is, $\Delta V_C = \frac{I_0 \alpha}{fC}$

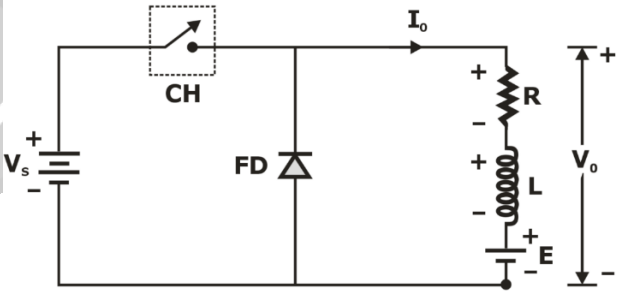
**Condition of continuous inductor current and capacitor voltage:**

If $V_C$ is the average capacitor voltage, the capacitor ripple voltage, $\Delta V_C = 2V_0$, which gives the critical value of the capacitor $C_C$ as $C_C = \frac{\alpha}{2fR}$.

If $I_L$ is average inductor current, the inductor ripple current $\Delta I = 2I_L$, which gives the critical value of the inductor $L_C$ as $L_C = L = \frac{(1-\alpha)^2 R}{2f}$

| Expression for $V_0$ | BUCK | BOOST | BUCK BOOST |
|---|---|---|---|
| In CCM | $V_0 = \alpha V_s$ | $V_0 = \frac{V_s}{1-\alpha}$ | $V_0 = -\frac{\alpha V_s}{1-\alpha}$ |
| In DCM | $V_0 = \frac{\alpha}{\beta} V_s$ | $V_0 = \frac{\beta V_s}{\beta - \alpha}$ | $V_0 = -\frac{\alpha V_s}{\beta - \alpha}$ |

**Steady State analysis of Type A Chopper:**




Average output voltage

$$V_0 = \alpha V_s$$

$V_{or} = \sqrt{\alpha}\, V_S$ (Rms value of output voltage)

$$I_{max} = \frac{V_S}{R}\left[\frac{1 - e^{-T_{on}/T_a}}{1 - e^{-T/T_a}}\right] - \frac{E}{R}$$

$$I_{min} = \frac{V_S}{R}\left[\frac{e^{T_{on}/T_a} - 1}{e^{T/Ta} - 1}\right] - \frac{E}{R}$$

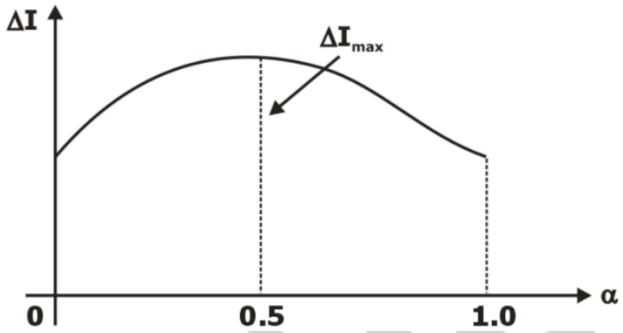
Where, $T_a$ = load time constant

$$T_a = \frac{L}{R}$$

Current ripple,

$$(\Delta I) = I_{max} - I_{min}$$

$$\Delta I = \frac{V_S}{R}\left[\frac{\left(1 - e^{-T_{on}/T_a}\right)\left(1 - e^{-T_{off}/T_a}\right)}{\left(1 - e^{-T/T_a}\right)}\right]$$




$T_{on} = \alpha T$

$T_{off} = (1 - ) \alpha T$

Per unit ripple (or) Ripple is a function of duty cycle 'α'. Ripple is minimum at α = 0, increases maximum at α = 0.5 and decrease at α = 1.0. For α = 0.5, ripple would be maximum.

$$(\Delta I)_{max} = \frac{V_S}{R}\left(\frac{\left(1 - e^{-0.5x}\right)\left(1 - e^{-0.5x}\right)}{1 - e^{-x}}\right) \qquad \left(\text{Let, } \frac{T}{T_a} = x\right)$$

$$(\Delta I)_{max} = \frac{V_S}{R}\tanh\left(\frac{R}{4fL}\right)$$

## 6. Inverters

**Series Inverters:** In a series inverter, the commutating elements L and C are connected in series with the load resistance R. The load resistance R can also be in parallel with C. The value of L and C are such that those form an underdamped circuit i.e.

$R^2 < \frac{4L}{C}$

$f = \left[\frac{1}{2\left(\frac{T}{2} + T_{off}\right)}\right]$ is the frequency of output voltage.

Where, $\frac{T}{2}$ is the time period of oscillations.

$T_{off}$ is the time gap between turn-off one thyristor and turn-on of the second thyristor.

$$\frac{T}{2} = \frac{\pi}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}}$$

The period of oscillation

**Bridge Inverter:** Bridge circuits are commonly used in DC-AC conversion. Moreover, an output transformer is not essential in a bridge circuit.

**1ϕ Half Bridge Inverter -** The output voltage volt $V_0 = \sum_{n-1,3,5...}^{\infty} \frac{2V_s}{n\pi} \sin n\omega t$

**1 ϕ Full Bridge Inverter-** The output voltage

$V_0 = \sum_{n-1,3,5,...}^{\infty} \frac{4V_s}{n\pi} \sin n\omega t$

Where, n = order of harmonic

$\omega = 2\pi f$, is frequency of the output voltage in red/sec

**Key points:**

- The load impedance ($Z_n$) is

$$Z_n = \left[R^{2+}\left(n\omega L - \frac{1}{n\omega C}\right)^2\right]^{1/2}$$

- Phase angle, $\phi_n = \tan^{-1}\frac{\left[n\omega L - \frac{1}{n\omega C}\right]}{R}$

**3phase Full Bridge VSI:**

| | $180^0$ Conduction | $120^0$ Conduction |
|---|---|---|
| Line Voltage RMS | $V_s\sqrt{\frac{2}{3}}$ | $V_s\frac{1}{\sqrt{2}}$ |
| Phase voltage RMS | $V_s\frac{\sqrt{2}}{3}$ | $V_s\frac{1}{\sqrt{6}}$ |
| Fundamental line voltage RMS | $V_s\frac{\sqrt{6}}{\pi}$ | $V_s\frac{3}{\pi\sqrt{2}}$ |
| Fundamental phase voltage RMS | $V_s\frac{\sqrt{2}}{\pi}$ | $V_s\frac{1}{\pi}\sqrt{\frac{3}{2}}$ |

**$180^0$ Conduction:**

1) Pole Voltages = $V_{A0} = \sum_{n=1,3,5...}^{\infty}\frac{2V_s}{n\pi}\sin n\omega t$

2) Line Voltages = $\sum_{n=1,3,5}\left(\frac{4V_s}{n\pi}\cos\left(n\frac{\pi}{6}\right)\right)\sin(n(\omega t+30^0))$

When n=3, 9, 15 Line voltage= 0, So Line voltages are free from Triplet harmonics

3) Phase Voltage= $\sum_{n=6k\pm1}\frac{2V_{dc}}{n\pi}\sin n\omega t$

$n = 6k \pm 1$ is due to stepped waveform

**$120^0$ Conduction:**

1) Pole and Phase Voltage are of same waveform

2) Triplet harmonics are absent in Phase and pole voltages

3) Line voltage contains $n = 6k \pm 1$ Harmonics

**Pulse Width Modulation:**

Let N= number of pulses per half cycle

Each pulse width = $\frac{2d}{N}$

Then Output voltage Expression is

$$V_0 = \sum_{n=1,3,5}\left(N\frac{4V_s}{n\pi}\sin n\gamma\sin\frac{nd}{N}\right)\sin n\omega t$$

Where $\gamma = \frac{\pi - 2d}{N+1} + \frac{d}{N}$

- Number of pulses per half cycle $N = \frac{f_c}{2f}$

  f= reference input frequency

$f_c$= Carrier input frequency

- Modulation Index $m_a = \frac{V_{Ref}}{V_{carrier}}$
- Relation between Pulse width and modulation index

$$\frac{2d}{N} = \frac{\pi}{N}(1 - m_a)$$

**Amplitude Modulation Depth:**

$$m_0 = \frac{\hat{V}_m}{\hat{V}_c}$$

Where $V_m$, $V_c$ are the modulating and carrier signal voltage peak values

For sinusoidal PWM, the amplitude modulation depth must be less than 1.0

**Output Voltages by Sinusoidal PWM:**

- In single phase half bridge VSI

  Fundamental peak output voltage = $\hat{V_{Ao1}} = m_a \frac{V_s}{2}$

- In single phase Full bridge VSI

  Fundamental peak output voltage = $\hat{V_{Ao1}} = m_a V_s$

- In Three phase Full bridge VSI

  Peak Fundamental Phase voltage $\hat{V}_{an1} = m_a \frac{V_{DC}}{2}$

  The fundamental line-line rms voltage is given by

$$V_{LLO_1} = \frac{\sqrt{3}}{2\sqrt{2}} m_a V_{DC}$$

- If peak value of carrier input and zero crossing of reference sinusoidal coincidence then, Number of Pulses per half cycle will be $N = \frac{f_c}{2f}$

  If Zero Crossing of carrier input and reference sinusoidal coincidence then, Number of Pulses per half cycle will be $N = \frac{f_c}{2f} - 1$

  If N is the number of pulses per half cycle then the predominant harmonics in the output is $2N \pm 1$

BYJU'S EXAM PREP

# Electrical Engineering

## Electrical and Electronic Measurements

SHORT NOTES

# IMPORTANT FORMULAS TO REMEMBER

## Chapter-1
## Error Analysis

Measurement is a process by which one can convert physical parameters to meaningful numbers. The measuring process is one in which the property of an object or system under consideration is compared to an accepted standard unit, a standard defined for that particular property.

**Static Characteristics**

1. **Accuracy**

   It is the closeness with which an instrument reading approaches the true value of the measured quantity.

2. **Precision**

   It is a measure of the reproducibility of the measurements. It is a measure of the degree of agreement within a group of measurements.

**Remember:**

- Precision is not the guarantee of accuracy.
- An instrument with a more significant figure has more precision.

3. **Sensitivity**

   It is the ratio of the magnitude of output signal to the magnitude of input signal applied to the instrument.

$$\text{Sensitivity} = \frac{\text{Output}}{\text{Input}}$$

**Note:**

- An instrument requires a high degree of sensitivity

$$\text{Sensitivity} \propto \frac{1}{\text{Deflection factor}}$$

4. **Resolution**

   The smallest change in input which can be detected with certainty by an instrument is its resolution.

5. **Linearity**

   The output is linearly proportional to the input. For a linear instrument, the sensitivity is constant for the entire instrument range. Linearity is the most important parameter compared to all other parameters.

**Remember:**

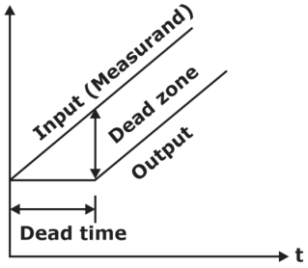
- Linearity is more important than sensitivity.
- Accuracy is more important than resolution.

**6. Dead Zone**

It is the largest change in input quantity for which there is no response from the instrument.

**7. Dead time**

Time required by an instrument to begin to respond to the change in a measurand.




**8. Range and Span**

The difference between the maximum and the minimum values of the scale is called range. The maximum value of the scale is called span.

**Errors**

Error= Measured value – True value = Accuracy

- Static Error

$\delta A = A_m - A_t$

Where,

$A_m$ = Measured value of a quantity or Actual value

$A_t$ = True value of a quantity or Nominal value

**Relative static error**

$$\epsilon_r = \frac{\delta A}{A_t}$$

**Static correction**

$$\delta C = A_t - A_m = -\delta A$$

**Static Sensitivity**

$$\text{Static sensitivity} = \frac{\Delta q_0}{\Delta q_i}$$

Where, $\Delta q0$ = Infinitesimal change in output, $\Delta qi$ = Infinitesimal change in input

**Non-linearity (N.L)**

$$\text{N.L.} = \frac{(\text{Max. deviation of output from the idealized straight line})}{\text{Full scale deflection}} \times 100$$

$$\text{Error at desired value} = \frac{\text{Full scale value} \times \text{Error at full scale}}{\text{Desired value}}$$

Combination of Quantities with Limiting Errors.Sum or Difference of Two or More than Two Quantities

Let $X = \pm X_1 \pm X_2 \pm X_3 \pm X_4$

Where,

$\pm \delta X_1$ = Relative increment in quantity $X_1$

$\pm \delta X_2$ = Relative increment in quantity $X_2$

$\pm \delta X$ = Relative increment in X

$\frac{\delta X_1}{X_1}$ = Relative limiting error in quantity $X_1$

$\frac{\delta X_2}{X_2}$ = Relative limiting error in quantity $X_2$

$\frac{\delta X}{X}$ = Relative limiting error in X

Product or Quotient of More than two Quantities

Let $X = X_1 X_2 X_3$ or $X = \frac{X_1}{X_2 X_3}$ or $X = \frac{1}{X_1 X_2 X_3}$

$$\frac{\delta X}{X} = \pm\left(\frac{\delta X_1}{X_1} + \frac{\delta X_2}{X_2} + \frac{\delta X_3}{X_3}\right)$$

**Composite Factors**

Let $X = X_1^n . X_2^m$

$$\frac{\delta X}{X} = \pm\left(n\frac{\delta X_1}{X_1} + m\frac{\delta X_2}{X_2}\right)$$

Arithmetic Mean

$$\bar{X} = \frac{\Sigma X}{n} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

where, $X_1, X_2, \ldots, X_n$= Readings or samples, n = Number of readings

**Deviation**

$d_n = x_n - \bar{X}$

**Note:** Algebraic sum of deviation is zero.

Average deviation, $\bar{D} = \frac{\Sigma |d|}{n} = \frac{|d_1| + |d_2| + \ldots |d_n|}{n}$

**Standard deviation**

for n > 20, $S.D. = \sigma = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{d_1^2 + d_2^2 + \ldots. + d_n^2}{n}}$

For n < 20 , $S.D. = s = \sqrt{\frac{\Sigma d^2}{n-1}}$

**Variance**

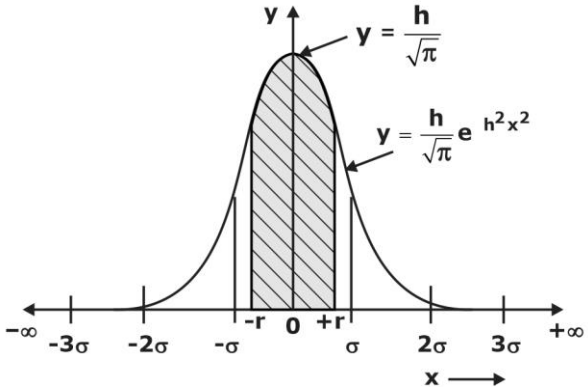
for n > 20, $v = \sigma^2 = \frac{\Sigma d^2}{n}$

For n < 20, $v = s^2 = \frac{\Sigma d^2}{n-1}$

**Normal or Gaussian Curve of Errors**

1. For Infinite Numbers of reading

$$y = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-x^2 / 2\sigma^2\right)$$




Where x = magnitude of deviation from mean, y = number of reading at any deviation x, (the probability of occurrence of deviation x)

σ = standard deviation

**Precision Index**

$$h = \frac{1}{\sigma\sqrt{2}}$$

**Probable error (P.E.)**

$$r = \frac{0.4769}{h}$$

**Average deviation**

$$D = \frac{r}{0.8453} = \frac{1}{\pi h^2}$$

**Standard deviation**

$$\sigma = \frac{r}{0.8745} = \frac{1}{h\sqrt{2}}$$

P.E. = r=0.8453 $\bar{D}$=0.6745σ

2. For Finite Numbers of Reading

For n > 20, P.E. = $r = 0.6745\sqrt{\frac{\Sigma |d|^2}{n}}$

For n <20, P.E. = $r = 0.6745\sqrt{\frac{\Sigma |d|^2}{n-1}}$

The standard deviation of the mean, $\sigma_m = \frac{\sigma}{\sqrt{n}}$

The standard deviation of standard deviation, $\sigma_\sigma = \frac{\sigma_m}{\sqrt{2}}$

The variance of combination of components

Let $x = f(X_1, X_2, \ldots\ldots X_n)$

$$V_x = \left(\frac{\partial X}{\partial x_1}\right)^2 V_{x_1} + \left(\frac{\partial X}{\partial X_2}\right)^2 V_{x_2} + .... + \left(\frac{\partial x}{\partial x_n}\right)^2 V_{x_n} \text{ where,}$$

$V_{x_1}, V_{x_2}, ....V_n = \text{variance of } x_1, x_2, ...x_n$

Standard Deviation of Combination of Components

Let x = f(X1, X2, ……, Xn)

$$\sigma_x = \left(\frac{\partial x}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial X}{\partial x_2}\right)^2 \sigma_{x_2}^2 + ....\left(\frac{\partial X}{\partial x_3}\right)^2 \sigma_{x_n}^2$$

where , $\sigma_{x_1}, \sigma_{x_2}, ....., \sigma_{x_n}$ = Standard deviation of X1, X2, ……, Xn

Probable Error of Combination of Components

Let X = f($X_1$, $X_2$, …., $X_n$)

$$r_x = \sqrt{\left(\frac{\partial x}{\partial x_1}\right)^2 r_{x_1}^2 + \left(\frac{\partial x}{\partial x_2}\right)^2 r_{x_2}^2 + ....\left(\frac{\partial x}{\partial x_n}\right)^2 r_{x_n}^2}$$

Where $r_{x_1}$, $r_{x_1}$……., $r_{x_n}$ = Probable error of

$X_1$, $X_2$,….., $X_n$

Uncertainty of Combination of Components

Let X = f($X_1$, $X_2$, …, $X_n$)

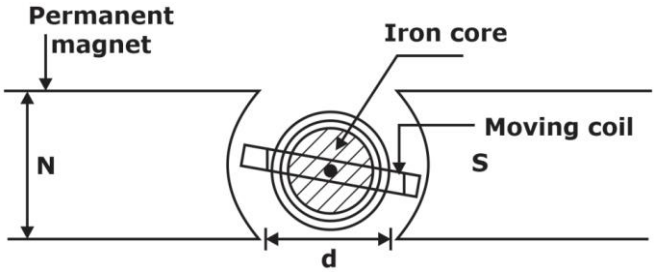
$$W_x = \sqrt{\left(\frac{\partial X}{\delta x_1}\right)^2 w_{x_1}^2 + \left(\frac{\partial X}{\partial x_2}\right)^2 w_{x_2}^2 + .....\left(\frac{\partial X}{\partial x_n}\right)^2 w_{x_n}^2}$$

Where $w_{x_1}, w_{x_2}, ...., w_{x_n}$ = Uncertainties of $X_1$, $X_2$, …, $X_n$

# Chapter-2

# Measurement of Current & Voltage

## D'Arsonval Galvanometer




### Deflecting torque

$T_d$ = BINA = GI

Where,

B = Flux density in air gap in $Wb/m^2$

I = current though moving coil in Ampere

N= Number of turns in coil

A = l.d = Area of coil in $m^2$

l, d = Length of vertical and horizontal side (width) of coil respectively in m

G = Displacement constant of galvanometer

### Controlling torque

$T_c = K\theta_f$

Where,

K = Spring constant of suspension; Nm/rad

$\theta_f$ = Final steady deflection of moving coil;

### Final steady deflection

$$\theta_f = \left(\frac{NBA}{K}\right) i = \left(\frac{G}{K}\right) i$$

### Dynamic behaviour of Torques in Galvanometers

Inertia torque, $T_j = J\frac{d^2\theta}{dt^2}$

Where J= moment of inertia of the moving system about the axis of rotation; $kg\text{-}m^2$,

θ = deflection at any time t; radian

Damping torque, $T_D = D\frac{d\theta}{dt}$

Where, D = damping constant

Controlling torque, $T_c = K\theta$ Where, K = control constant

Deflecting Torque, $T_d$ = GI

Equation of motion, $T_j + T_D + T_C = Td$

$$J\frac{d^2\theta}{dt^2}+\frac{Dd\theta}{dt}+K\theta=GI$$

**Note:**

If $D^2 < 4$ KJ, galvanometer is under-damped.

If $D^2 = 4$ KJ, galvanometer is critically damped.

If $D^2 > 4$ KJ, galvanometer is over-damped.

The total resistance of galvanometer circuit for critical damping, $R=\frac{G^2}{2\sqrt{KJ}}$

External series resistance is required for critical damping, $R_e=R-R_g=\frac{G^2}{2\sqrt{KJ}}-R_g$

Where $R_g$ = Resistance of galvanometer

**Sensitivity**

Current sensitivity

$$S_i=\frac{\theta_f}{i}=\frac{G}{k}\text{ rad/A}=\frac{d}{1\times10^6}\text{ scale divisions / }\mu\text{A}=\frac{2000G}{K\times10^6}\text{ mm / }\mu\text{A}$$

Voltage sensitivity

$$S_i=\frac{d}{iR_g\times10^6}\text{ scale division / }\mu\text{V}$$

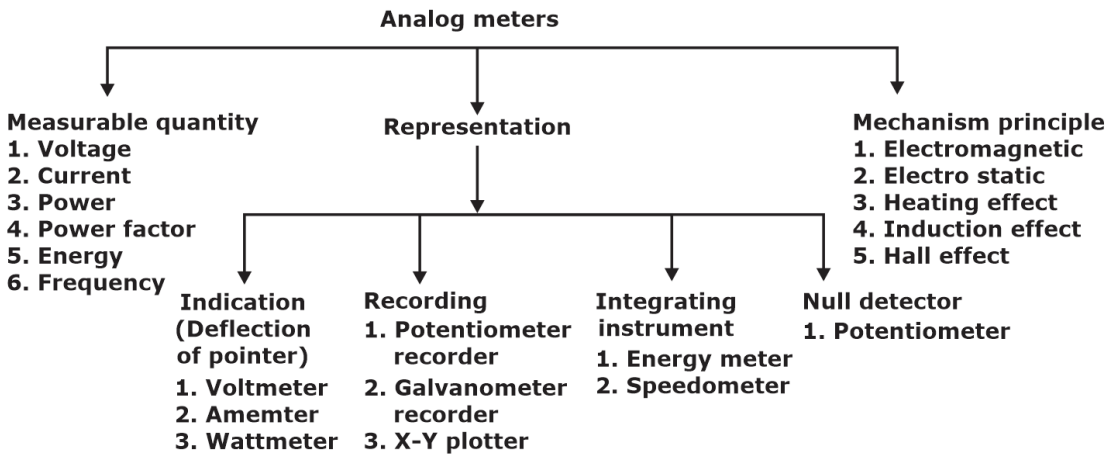
Mega ohm sensitivity

$$S_o=\frac{d}{i\times10^{-6}}\text{ M}\Omega\text{ / scale division}$$

**Remember:**

The sensitive galvanometer produces a large deflection for a small current.

**Classification of Analog Meters**




**Torque in Analog Meter**

1. Deflecting Torque ($T_D$) is proportional to quantity under measurement. This torque deflects the pointer away from the initial or zero position.

$T_D \propto$ measurable quantity

2. Controlling Torque ($T_C$)

   The controlling torque is opposite to deflecting torque. When deflecting torque equal to controlling torque, the pointer comes to the final steady state position.

   At equilibrium, $T_C = T_D$

**Note :**

Control torque is also used to bring the pointer to zero from the initial position if there is no deflecting torque.

Except in PMMC, in all other instruments, if the control spring fails or is broken, the pointer moves to the maximum scale position.

Control torque is provided by

(i) Spring control, (ii) Gravity control

**3. Damping Torque**

It is used to damp out oscillation at the final steady state position. The time response of the instrument depends on damping torque.

Damping torque provided by:

(i) Air friction damping; Used where low magnetic fields are produced.

(ii) Fluid friction damping; Used where deflecting torque is minimum.

(iii) Eddy current damping; Used where permanent magnet produces the required deflecting torque.

**Error in Analog Meters**

**1. Frictional error**

To reduce the frictional error, the torque to weight ratio of the instrument should be high.

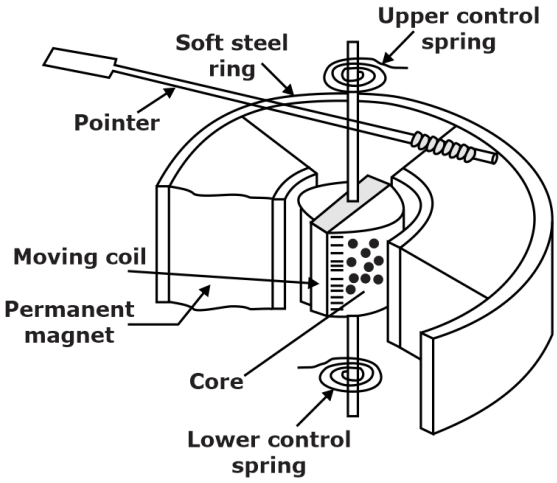
**2. Temperature error**

Due to temperature change, change in resistance of meters and shunts and series multiplier occurs. To reduce this effect, resistances are made up of manganin material.

**3. Frequency error**

Due to change in frequency, error is produced in instrument because change in frequency causes a change in reactance. To reduce this error, a capacitance is used in case of voltmeter and for ammeter, the time constant and shunt impedance are maintained at same value.

**Permanent Magnet Moving Coil (PMMC)**




**Deflection torque**

$T_D$ =nBAI, $T_D$ = GI,

Where, G = nBA, n = Number of turns, B = Flux density, A = Area of core

I = Current to be measured

Final steady-state deflection, $\theta = \left(\frac{G}{k}\right)I$

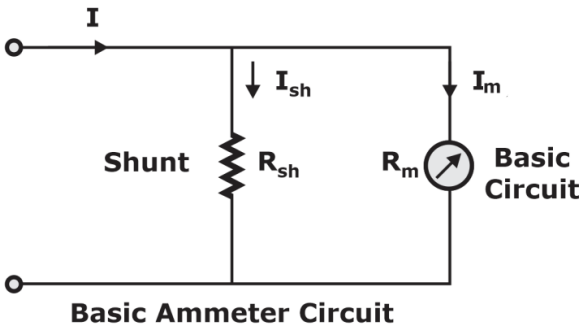
Where, K = Spring constant

**Note:**

- PMMC instrument measures only DC or average values.
- Scale is linear.
- Spring is used for controlling torque.
- Damping torque provided by eddy current damping.
- It has a high torque to weight ratio, so accuracy and sensitivity are higher than other instruments.
- In direct measurement, the PMMC measures up to a current of 50 mA or a voltage of 100 mV without any external device.

**Enhancement of Ammeters and Voltmeters**

**1. Ammeter Shunts**




Basic Ammeter Circuit

$$I_{sh}R_{sh} = I_mR_m$$

$$I = \left(1 + \frac{R_m}{R_{sh}}\right)I_m$$

where, I = Current to be measured; $I_m = I_{fs}$ = Full scale deflection current; A

$R_m$ = Internal resistance of meter; Ω, $R_{sh}$ = Resistance of the shunt; Ω

**Shunt resistance**

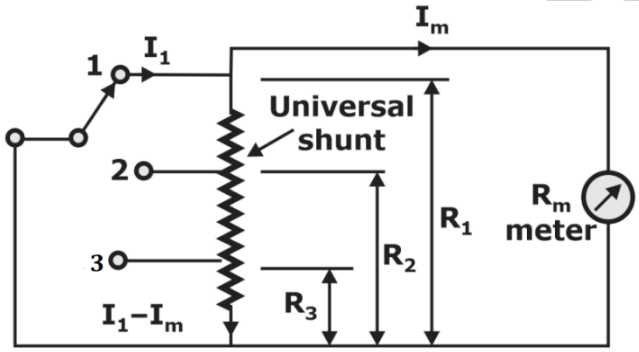
$$R_{sh} = \frac{R_m}{m-1}$$

Where, $m = \frac{I}{I_m} = 1 + \frac{R_m}{R_{sh}}$ , m = Multiplying factor for shunt

**Note:**

To reduce the temperature effect, swamp resistance made up of manganin is added in series with an ammeter.

**2. Universal or Ayrton Shunt**




**(Multi-range ammeter using universal shunt)**

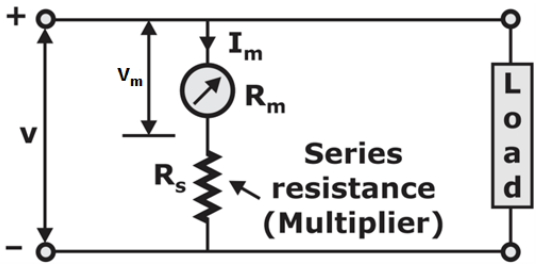
For switch at a position 1, $R_1 = \frac{R_m}{(m-1)}$

For switch at a position 2, $R_2 = \frac{(R_1 + R_m)}{m_2}$

For switch at a position 3, $R_3 = \frac{(R_1 + R_m)}{m_3}$

Where, $m_1 = \frac{I_1}{I_m}$, $m_2 = \frac{I_2}{m}$, $m_3 = \frac{I_3}{I_m}$

**3. Voltmeter Multipliers**

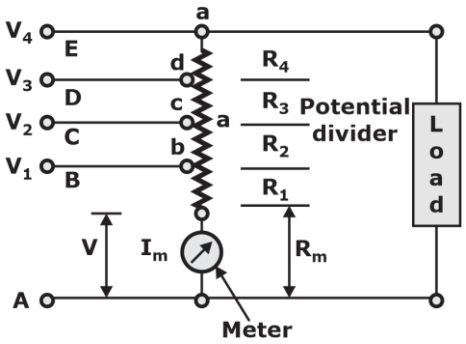



The multiplying factor for the multiplier, $m = \frac{V}{V_m} = 1 + \frac{R_s}{R_m}$

Resistance of multiplier, $R_s = (m-1)R_m$

Where, $R_s$ = Multiplier resistance, $R_m$ = Internal resistance of meter

### 4. Potential Divider Arrangement




$R_1 = (m-1)R_m,\ R_2 = (m_2 - m_1)R_m, R_3 = (m_3 - m_2)R_m, R_4 = (m_4 - m_3)R_m$

Where, $R_1$ = Resistance between point a and b, $R_2$ = Resistance between point b and c

$R_3$ = Resistance between point a and d $R_4$ = Resistance between point d and e

Voltmeter Sensitivity (VS), $S_v = \frac{1}{I_{fs}} = \frac{R_s + R_m}{V}\ \Omega/V$

**Remember:**

To reduce loading effect, a voltmeter with higher value of sensitivity is preferred.

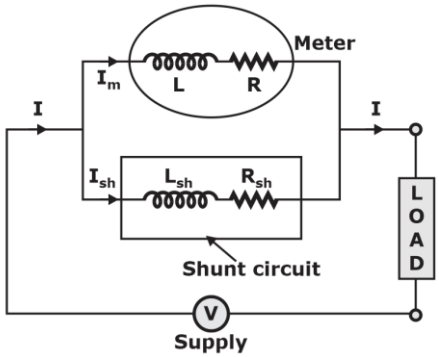
### Moving Iron Instruments

Deflecting torque, $T_d = \frac{1}{2}I^2\frac{dL}{d\theta}$, Deflection, $\theta = \frac{1}{2}\frac{I^2}{K}\frac{dL}{d\theta}$

**Note:**

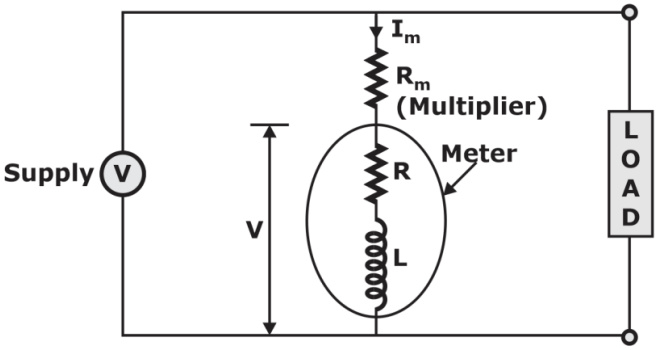
- Moving iron instrument measure both A.C and D.C. quantities.
- In case of A.C, it measures RMS Value.
- Scale is nonlinear.
- Controlling torque is provided by spring, and air friction damping is used.
- Curve between $\frac{dL}{d\theta}$ and θ is a rectangular hyperbola.

### Shunts for Moving Iron Instruments



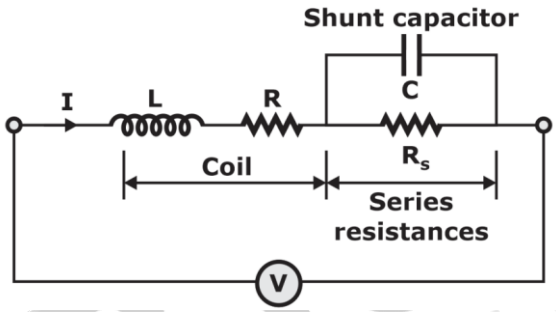

$$\frac{I_{sh}}{I_m} = \frac{R}{R_{sh}}\frac{\sqrt{1+(\omega L/R)^2}}{\sqrt{1+(\omega L_{sh}/R_{sh})^2}}$$

**Multipliers for Moving Iron Instruments**




Voltage multiplying factor, $m = \frac{V}{v} = \frac{\sqrt{(R + R_m)^2 + \omega^2 L^2}}{\sqrt{R^2 + \omega^2 L^2}}$

**Error in Moving Iron Instruments**




**Shunt capacitance**

$$C = 0.41\frac{L}{R_s^2}$$

**Eddy currents**

When ω is small, $I_e' = \frac{\omega^2 M L_e I}{R_e^2}$

When ω is large, $I_e' = \frac{M L_e I}{R_e} = \text{constant}$

Where,

$R_e$, $L_e$ = resistance and inductance of eddy current path

**Note:**

- Moving iron instrument is not suitable for measurement of current or voltage for frequency above 125 Hz because eddy current is constant at a higher frequency.
- If meter time constant is equal to the shunt time constant then ammeter is made independent of the input supply frequency.
- To reduce hysteresis error, the iron part of moving iron is made up of Nickel iron alloy.
- To reduce the external stray magnetic fields.

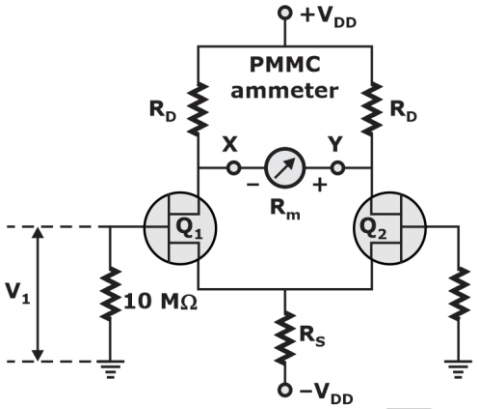
**Average current through diode vacuum tube voltmeters**

$$I_{av} = \frac{E_{av}}{2R} = \frac{E_{rms}}{2 \times 1.11 \times R} = 0.45\frac{E_{rms}}{R}$$

Where, $E_{rms}$ = RMS value of applied voltage, $E_{av}$ = Average value of applied voltage

R = Load resistance

**Difference amplifier type of electronic voltmeter**




Thevenin's voltage across terminal X-Y

$$V_{Th} = g_m \left(\frac{r_d R_D}{r_d + R_D}\right) v_1$$

Thevenin's resistance looking into terminals X-Y

$$R_{Th} = \frac{2r_d R_D}{r_d + R_D}$$

Where, $R_d$ = A.C. drain resistance in Ω, $g_m$ = Transconductance in mho

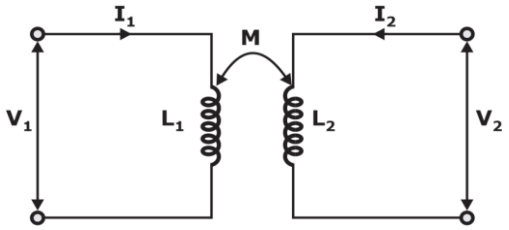
Current through ammeter, $i = \frac{V_{Th}}{R_{Th} + R_m} = \frac{g_m r_d R_D / (r_d + R_D)}{2r_d R_D / (r_d + R_D) + R_m} . v_1$

Where, $R_D << r_d$, $i = \frac{g_m R_D}{2R_D + R_m} . v_1$

# Chapter-3

# Measurement of Power & Energy

## Electrodynamometer




(a) If $i_1$ and $i_2$ are D.C current i.e. $i_1 = i_2 = I$, $T_d = I^2 \frac{dM}{d\theta}$ (Measure average value)

(b) If $i_1$ and $i_2$ are A.C current and no phase shift, $i_1 = i_2 = I$, $T_d = I^2 \frac{dM}{d\theta}$ (Measure RMS value)

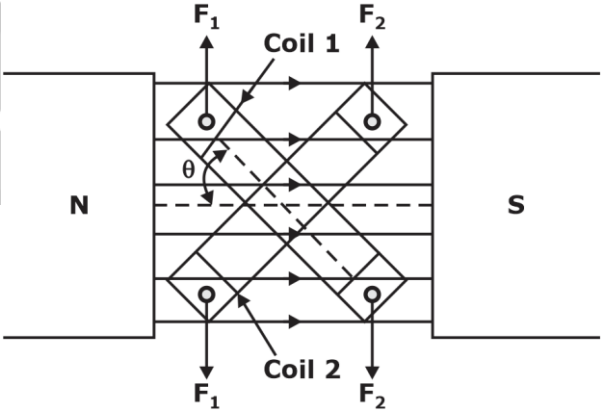
(c) If $i_1 = I_{m1} \sin \omega t$ and $i_2 = i_{m2} \sin (\omega t - \phi)$, $T_d = I_1 I_2 \cos\phi \frac{dM}{d\theta}$ (Measure RMS value)

where, $I_1 = \frac{I_{m1}}{\sqrt{2}}$ and $I_2 = \frac{I_{m2}}{\sqrt{2}}$

**Note:**

- Electrodynamometer instrument is a transfer instrument
- It measures both A.C. and D.C
- Scale is nonlinear.
- Its sensitivity is lesser than PMMC and M.I. type instrument.

## Ratiometer




Deflecting torque acting on coil 1, $T_{d1} = N_1 B I_1 d_1 I_1 \cos\theta$

Deflecting torque acting on coil 2, $T_{d2} = N_2 B I_2 d_2 I_2 \cos\theta$

Where

$I_1$, $I_2$ = current in coil 1 and 2, $N_1$, $N_2$ = Number of turns in coil 1 and 2

$l_1$, $l_2$ = length of coil 1 and 2, $d_1$, $d_2$ = width of coil 1 and 2

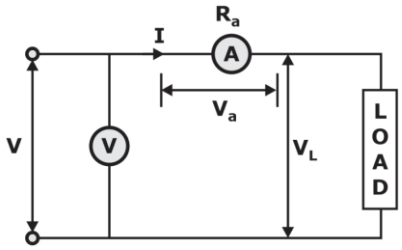
B = flux density of magnetic field

Deflection at equilibrium, $\theta = k\left(\frac{I_1}{I_2}\right)$

**Measurement of Power**

**1. D.C Circuits**

Ammeter connected between load, and voltmeter

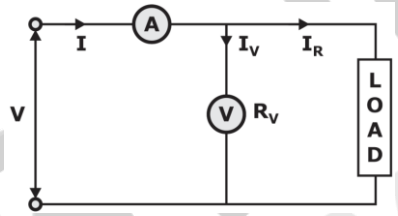



Power consumed by load :

$P = VI - I^2R_a$

Where, V= Voltage across voltmeter, I = Current though ammeter, $R_a$ = Resistance of ammeter

A voltmeter connected between the load and ammeter




Power consumed by load :

$P = VI - \frac{V^2}{R_v}$

Where, V = Voltage across voltmeter, I = Current through ammeter,

RV = Resistance of ammeter

2. A.C. Circuits

Instantaneous power

$P = vi = V_m I_m \sin\omega t \sin(\omega t - \phi)$

Where $v = V_m \sin\omega t$, $i = I_m \sin(\omega t - \phi)$

Average power, $P = VI\cos\phi = \frac{V_m I_m}{2}\cos\phi$

Where V, I = RMS values of voltage and current, cosϕ = Power factor of the load

Let

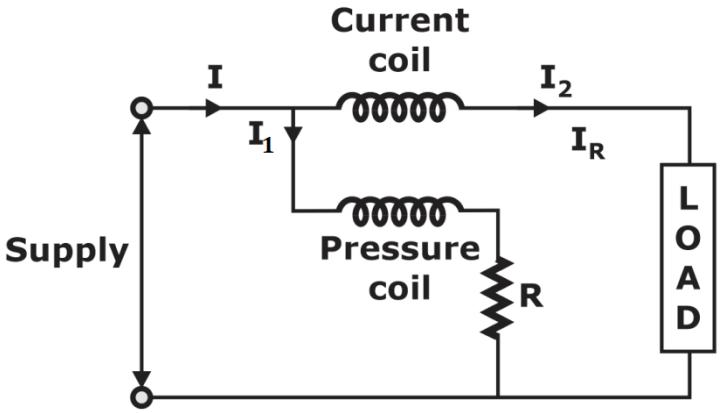
$v = V_o + \sum_{n=1}^{m} V_n \sin(n\omega t + \theta_n)$ and $i = I_o \sum_{n=1}^{m} I_n \sin(n\omega t + \phi_n)$ then

$p_{avg} = V_o I_o + \frac{1}{2}\sum_{n=1}^{m} V_n I_n \cos[\theta_n - \phi_n]$

**Remember:**

Wattmeter reads average active power.

**Electrodynamometer Wattmeter**




**Instantaneous torque**

$$T_i = i_1 i_2 \left(\frac{dM}{d\theta}\right)$$

where $i_1$, $i_2$ = instantaneous value of current in pressure and current coils

**Deflecting torque**

$$T_d = \frac{VI}{R_p} \cos\phi . \frac{dM}{d\theta}$$

Where $R_p$ = resistance of pressure coil circuit

**Controlling torque**

$$T_c = K\theta$$

Where, K = spring constant, θ = final steady deflection

Deflection, $\theta = \left(K_1 \frac{dM}{d\theta}\right) P$

Where, P = Power being measured = Vl cos$\phi$, $K_1 = \frac{1}{R_p K}$

**Note:**

Scale is linear in terms of power as $\theta \propto P$

**Errors in Electrodynamometer Wattmeter Correction factor (K)**

The correction factor is multiplied by the actual wattmeter reading to get the true power.

For lagging power factor, $K = \frac{\cos\phi}{\cos\beta \cos(\phi - \beta)}$

For leading power factor, $K = \frac{\cos\phi}{\cos\beta \cos(\phi + \beta)}$

Where,

$\phi$ = Angle between current in the current coil and voltage of pressure coil

β = Angle between current and voltage of pressure coil

True power = Correction factor x actual wattmeter reading

**For Very small β**

Actual wattmeter reading = true power (1 + tan $\phi$ tan β)

Error = tan φ tan β × true power = VI sin $\phi$ tan β

%error = tan $\phi$ tan $\beta \times 100$

True power = VI cos $\phi$

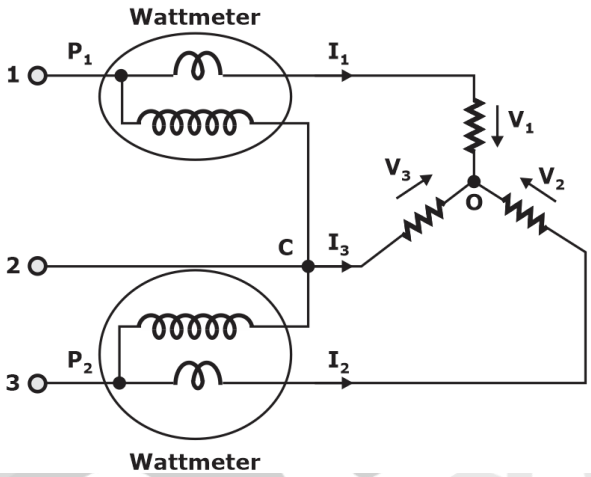
Where, V = Voltage applied to pressure coil, I = Current in current coil

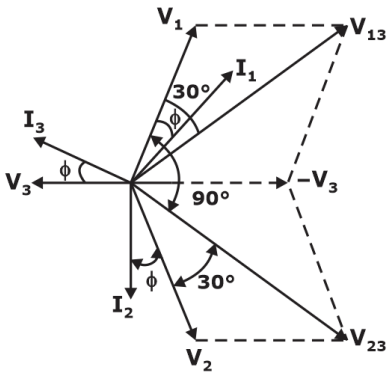
**Power in Poly-Phase Systems**

**Blondel's Theorem**

Suppose a network is supplied through n conductors. In that case, the total power is measured by summing the reading of n wattmeters so arranged that a current element of a wattmeter is in each line and the corresponding voltage element is connected between that line and a common point, if the common point is located on one of the lines, then the power may be measured by (n – 1) wattmeters.

**Two wattmeter method**




Reading of $P_1$ wattmeter, $P_1 = \sqrt{3}\ VI\cos(30^\circ - \varphi)$

Reading of $P_2$ wattmeter, $P_2 = \sqrt{3}\ VI\cos(30^\circ + \varphi)$

Total power consumed by load, $P = P_1 + P_2$

Power factor, $\cos\varphi = \cos\left[\tan^{-1}\left(\sqrt{3}\frac{P_1 - P_2}{P_1 + P_2}\right)\right]$

$$\tan\varphi = \sqrt{3}\frac{P_1 - P_2}{P_1 + P_2}$$

Where, V = Phase voltage, I = Phase current

$\varphi$ = Angle between phase current and phase voltage

Reading of wattmeter at Different Power Factor

| S.No. | $\varphi$ | cos $\varphi$ | $P_1$ | $P_2$ | $P=P_1 + P_2$ | Comment |
|---|---|---|---|---|---|---|
| 1. | 0 | 1 | $\frac{\sqrt{3}}{2} V_L I_L$ | $\frac{\sqrt{3}}{2} V_L I_L$ | $\sqrt{3}\ V_L I_L$ | $P_1 = P_2$ (equal reading) |
| 2. | 30° | 0.866 | $V_L I_L$ | $\frac{V_L I_L}{2}$ | $1.5\ V_L I_L$ | $P_1=2P_2$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| 3. | 60° | 0.5 | $\frac{\sqrt{3}}{2} V_L I_L$ | 0 | $\frac{\sqrt{3}}{2} V_L I_L$ | $P_2=0$, $P_1=P$ |
| 4. | 90° | 0 | $+\frac{V_L I_L}{2}$ | $-\frac{V_L I_L}{2}$ | 0 | $P_1 = -P_2$ |

**Note:**

When Wattmeter reading comes in negative, reverse either P.C. or C.C. terminal and take the reading of the negative wattmeter.

**Measurement of Energy**

For the measurement of energy, we use an energy meter. An energy meter is an integrating instrument that adds energy cumulatively over time.

$$\text{Energy} = \text{Power} \times \text{time} = \int_0^t P \, . \, dt \text{ kWhr}$$

**Note**:

An energy meter works on principle of induction.

- The meter which measures A.C. energy is called a watt-hour meter.
- The meter which measures D.C. energy is called an amp-hour meter.

Deflection torque, $T_D \propto P$

Breaking torque, $T_B \propto N$

Where, N = Speed of disc in rps

At balance, $T_d = T_B$

$$\int P \, . \, dt = K \int N \, . dt$$

$$\text{Energy} \propto \int N \, . dt$$

Energy meter constant (EMC)

$$\text{EMC} = \frac{\text{Number of revolution made by disc}}{\text{Energy recorded in kWhr.}}$$

Where P = Power in kW, T = Time in hrs.

Revolution of disc due, $\%\text{Creeping Error} = \frac{\text{creeping per hour}}{\text{Revolution of disc due to total load per hour}} \times 100$

**Remember:**

Potential coil of energy meter should be highly inductive so that it measures true energy.

**Compensation in Energy meter**

1. Lag compensation: Through lag coil or shading coil.

2. Low load or friction adjustment: By using shading loop.
3. Over friction or creeping: By providing holes or slots on rotating disc.
4. Overload compensation: By keeping saturable shunt magnet in series magnet or current coil.
5. Overvoltage compensation: By keeping saturable shunt magnet in shunt magnet.
6. Speed adjustment: By adjusting position of break magnet.

**Remember:**

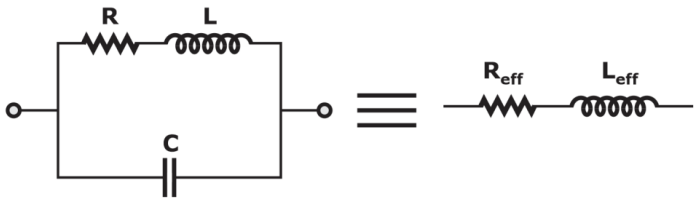
Creeping error is always positive.

If either potential coil or current coil is wrongly connected then the disc rotates in opposite direction.

# Chapter-4

# Measurement of R, L & C (Resistors, Inductors, Capacitors)

## Frequency Errors in Resistors




(Equivalent circuit of a resistor at low and medium frequencies)

**Effective resistance**

$$R_{eff} = \frac{R}{1+\omega^2 C\left(CR^2 - 2L\right)}$$

Effective inductance or residual inductance

$$L_{eff} = \frac{L - CR^2}{1+\omega^2 C\left(CR^2 - 2L\right)}$$

$$\tan\phi = \frac{X_{eff}}{R_{eff}} = \frac{\omega L_{eff}}{R_{eff}} = \frac{\omega\left(L - CR^2\right)}{R} = \omega\left(\frac{L}{R} - CR\right)$$

Where, φ = Phase deflection angle Time constant

Condition for resistance to remain independent of frequency, $CR^2 = 2L$

Condition for resistance to show no inductive effect, $CR^2 = L$

Effective resistance for zero effective inductance, $R_{eff} = \frac{R}{1-\omega^2 LC}$

Quality factor, $Q = \frac{\omega L}{R}$

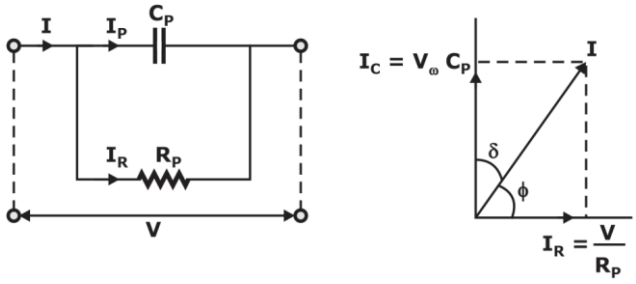
## Frequency Error in Inductors

Effective resistance, $R_{eff} = \frac{R}{\left(1-\omega^2 LC\right)}$

Effective inductance, $L_{eff} = L\left(1+\omega^2 LC\right)$

## Capacitor

1. Parallel Representation

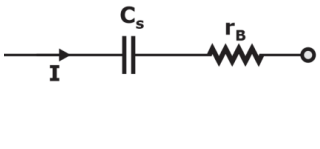


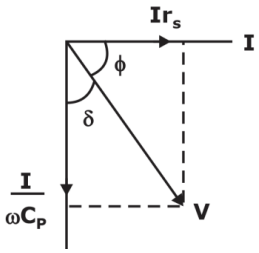
Dielectric loss, $P_L = \omega C_p V^2 \tan\delta$

Dissipation factor, $D = \tan\delta = \frac{1}{\omega C_p R_p}$

Where, δ = loss angle of the capacitor.

2. Series Representation

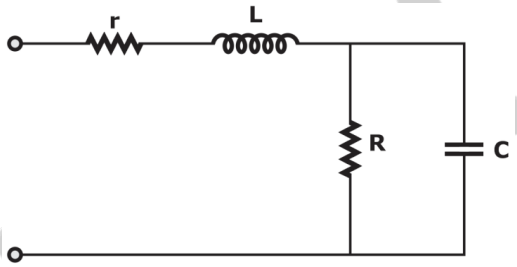



Dissipation factor, $P_L = \frac{I^2}{\omega C_s} \tan\delta$

Dissipation factor, $D = \tan\delta = \omega C_s r_s$

**FREQUENCY Errors in Capacitors**




Equivalent circuit of a capacitor

Equivalent Circuit of a Capacitor

Effective capacitance, $C_{eff} = \frac{C}{1 - \omega^2 LC}$

**1. For Medium Frequency**

Effective capacitance, $C_{eff} = C(1 + \omega^2 LC)$

Effective series resistance, $R_{eff} = r + \frac{R}{1 + \omega^2 R^2 C^2}$

Where, r = resistance of load

Loss angle, $\tan\delta = \frac{1 - \omega^2 LC}{\omega r + \frac{1}{\omega CR}}$

**2. For Low Frequency**

Effective capacitance, $C_{eff} = C + \frac{1}{\omega^2 CR^2}$

Effective series resistance, $R_{eff} = \frac{R}{1 + \omega^2 C^2 R^2}$

Loss angle, $\tan\delta = \frac{1}{\omega CR}$

**Classification of Resistance**

1. Low resistance: All resistance of the order of 1 Ω and below.
   Example: Winding coils of electrical motors, generators and transformers.
2. Medium resistance: Resistances from 1 Ω upwards to about 0.1 MΩ.
   Example: Resistance of heaters, potentiometers.
3. High resistance: All resistances of the order of 0.1 MΩ and above.
   Example: Insulation of electrical cable and windings, insulation of motors, generators and transformers.

**Measurement of Medium Resistance**

The different methods employed are:

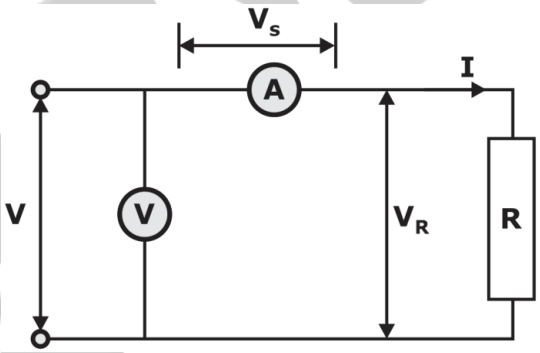
(i) Ammeter – voltmeter method
(ii) Wheatstone bridge method
(iii) Ohmmeter method
(iv) Substitution method

**1. Ammeter Voltmeter Method**

$$R_m = \frac{\text{Voltmeter reading}}{\text{Ammeter reading}} = \frac{V}{I}$$

Where $R_m$ = measured value of resistance

(a) Circuit for higher resistance




The true value of resistance, $R = R_{m1} - R_a$

$$R_m = R_{m1}\left(1 - \frac{R_a}{R_{m1}}\right)$$

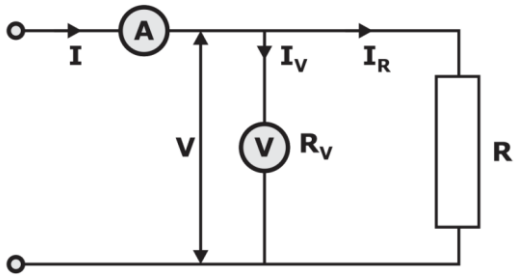
Where, $R_{m1}$ = Measured value of resistance, $R_a$ = Resistance of ammeter

**Relative error**

$$\epsilon_r = \frac{R_{m_1} - R}{R} = \frac{R_a}{R}$$

To get minimum error, the test resistance should be more than the ammeter resistance so that this adjustment is suitable for measurement of high resistance.

(b) Circuit for lower resistance




**The true value of resistance**

$$R = \frac{R_{m2}R_v}{R_v - R_{m2}}$$

Where, $R_{m2}$ = Measured value of resistance, $R_v$ = Resistance of voltmeter

For $R_v >> R_{m2}$

$$R = R_{m2}\left(1 - \frac{R_{m2}}{R_v}\right)$$

Relative error, $\epsilon_r = \frac{R_{m_2} - R}{R} = \frac{-R^2_{m2}}{R_V R} \quad \left[R_V >> R_{m_2}\right]$

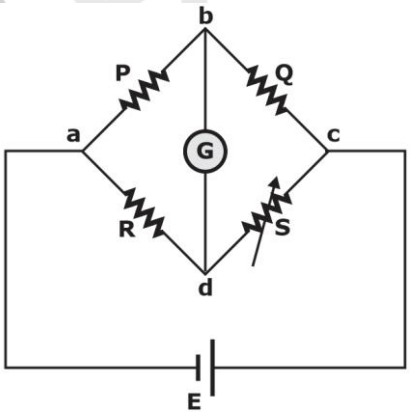
Approximate relative error, $\epsilon = -\frac{R}{R_V} \quad \left[\text{for} R_{m_2} \approx R\right]$

**Note:**

Relative errors for the above two cases are equal when true value of resistance,

$R = \sqrt{R_a R_V}$

## 2. Wheatstone Bridge




At Balance, $R = S\frac{P}{Q}$

Sensitivity of Wheatstone bridge, $S_B = \frac{\theta}{\Delta R / R} = \frac{S_v ESR}{(R+S)^2}; mm = \frac{S_v E}{\frac{P}{Q} + 2 + \frac{Q}{P}}$

Where,

$S_v$ = Voltage sensitivity of galvanometer, mm/volt

E = Bridge voltage

P, Q = Branch resistances

θ = Deflection of galvanometer, mm

For a bridge with equal arms, $S_B = \frac{S_V E}{4}$

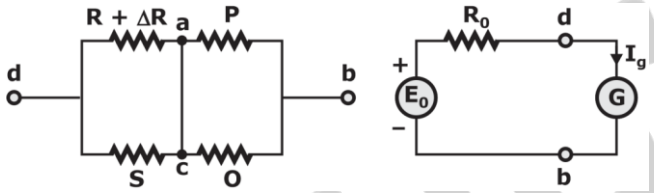
Note:

- For maximum bridge sensitivity

$$\frac{P}{Q} = \frac{R}{S} = 1$$

- Sensitivity of bridge is most important parameter as compared to accuracy, precision and resolution.

**Equivalent circuit of Wheatstone bridge**




Galvanometer current, $I_g = \frac{E_0}{R_0 + G}$

Where,

$E_0$ = Thevenin's or open circuit voltage appearing between terminals b and d with galvanometer circuit open circuited.

G = Resistance of the galvanometer circuit

$$E_0 = E\left[\frac{R + \Delta R}{2R + \Delta R} - \frac{1}{2}\right] \approx E\left(\frac{\Delta R}{4R}\right) \text{ as DR << R}$$

ΔR = Change in resistance R

Thevenin equivalent resistance of bridge, $R_0 = \frac{RS}{R + S} + \frac{PQ}{P + Q}$

Galvanometer deflection, $\theta = \frac{S_V E S \Delta R}{(R + S)^2} = \frac{S_i E S \Delta R}{(R_0 + G)(R + S)^2}$

Where,

$S_i$ = Current sensitivity of galvanometer Bridge sensitivity

$$S_B = \frac{\theta}{\Delta R / R} = \frac{S_i E S R}{(R_0 + G)(R + S)^2}$$

Current sensitivity

$$S_i = \frac{\theta}{I_g} \text{ ; mm/μA}$$

θ = Deflection in the galvanometer

$I_g$ = Current in galvanometer voltage sensitivity

$$S_V = \frac{\theta}{V_{Th}} \text{ ; mm/V}$$

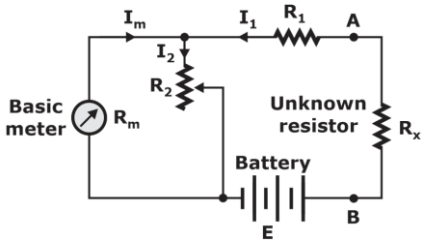
$V_{Th}$ = Voltage across galvanometer

**Note:**

In the Wheatstone bridge method, the effect of lead resistance is not eliminated; hence, it is unsuitable for measuring low resistance.

**3. Ohmmeters**

(a) Series Type ohmmeter



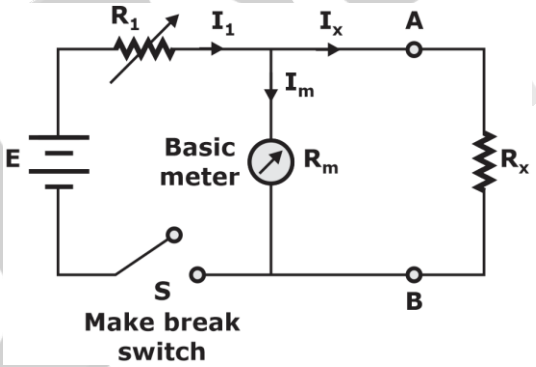

Half scale resistance, $R_h = R_1 + \frac{R_2 R_m}{R_2 + R_m}$

Meter current, $I_m = \frac{ER_2}{(R_h + R_X)(R_2 + R_m)}$

Full-scale deflection current, $I_{fs} = \frac{ER_2}{R_h(R_2 + R_m)}$

The friction of full-scale reading, $S = \frac{I_m}{I_{fs}} = \frac{R_m}{R_x + R_h}$

(b) Shunt Type Ohmmeter




Half-scale reading of unknown resistance $R_x$ is

$$R_h = \frac{R_1 R_m}{R_1 + R_m}$$

Half scale reading of the meter, $I_h = 0.5 I_{fs} = \frac{ER_h}{R_1 R_m + R_h(R_1 + R_m)}$

Where, $R_m$ = Internal resistance of meter, $R_1$ = Adjustable resistor (as shown in figure)

E = Supply voltage

**Measurement of low resistance**

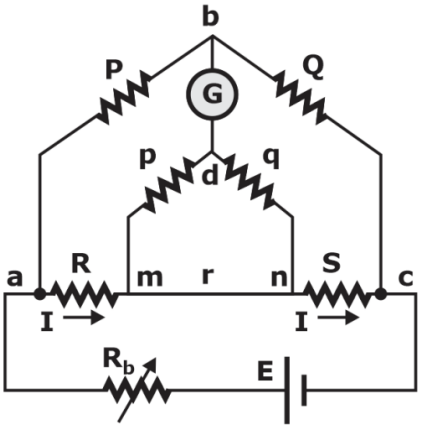
The different methods employed are:

(i) Kelvin's double bridge method

(ii) Ammeter voltmeter method

(iii) Potentiometer method

**Kelvin's Double Bridge Method**




For zero galvanometer deflection, $E_{ab} = E_{amd}$

$$R = \frac{P}{Q}.S + \frac{qr}{p+q+r}\left[\frac{P}{Q} - \frac{p}{q}\right]$$

If $\frac{P}{Q} = \frac{p}{q}$

Then $R = \frac{P}{Q}.S$

**Note:**

Accuracies by Kelvin double bridge method

(i) From 1000 μΩ to 1.0 Ω: 0.005%, (ii) From 100 μΩ to 1000 μΩ: 0.05% to 0.05%

(iii) From 10 μΩ to 100 μΩ: 0.5% to 0.2%

**Measurement of High Resistance**

The different methods employed are:

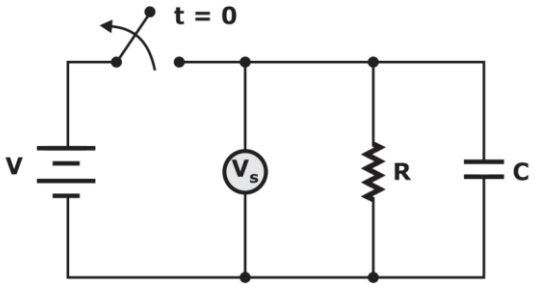
(i) Loss of charge method

(ii) Megger

(iii) Direct deflection method

(iv) Mega ohm bridge

**1. Loss of charge Method**



$$R = \frac{0.4343t}{C\log_{10}\left(\frac{V}{V_C}\right)}$$

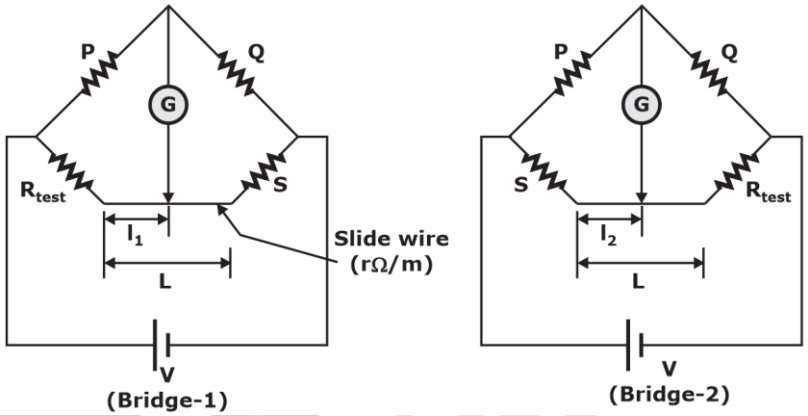
**2. Megger**

- Megger works on the principle of electrodynamometer.
- Megger is used to measure cable, motor, generator insulation resistance, etc.
- Deflecting torque angle is proportional to the resistance of the insulator, which is under test.
- It is also used to check the continuity of cable.
- No external control torque was provided. Air friction damping is used.
- No need of an external power supply.

**Note:**

High resistance have a guard terminal which is used to avoid leakage current.

**Carry Foster Slide Wire Bridge**




From bridge (1), $\frac{P}{Q}+1=\frac{R+S+Lr}{S+(L-l_1)r}$ ...(i)

From bridge (2), $\frac{P}{Q}+1=\frac{R+S+Lr}{R+(L-l_2)r}$ ...(ii)

Equating equation (i) and (ii)

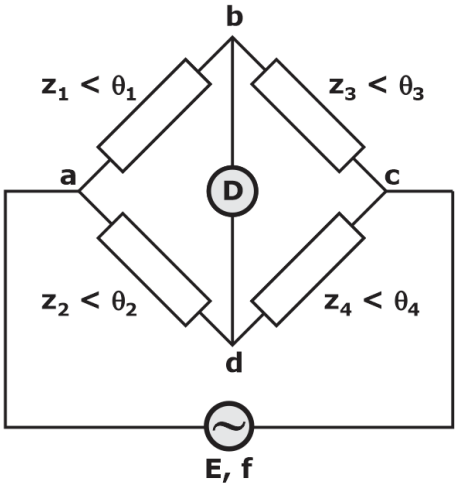
$R - S = (l_2 - l_1)r$

**Note:**

Carry Foster bridge method is used for medium resistance measurement compared with standard resistance.

## A.C. Bridges:-

Used to measure self-inductance, Mutual inductance, capacitance, and frequency.




General Equation for bridge balance, $\bar{z}_1\bar{z}_4 = \bar{z}_2\bar{z}_3$

Magnitude condition, $|z_1|\,|z_4| = |z_2|\,|z_3|$

Angle condition, $\angle\theta_1 + \angle\theta_4 = \angle\theta_2 + \angle\theta_3$

**Note:**

Magnitude and angle conditions must be satisfied for the bridge to be balanced.

Depending upon the frequency, different null detectors are used.

Vibration galvanometer - 5 Hz to 1 kHz

Headphones - 250 Hz to 4 kHz

Tuned amplifier detector - 10 Hz to 100 kHz

D'Arsonval Galvanometer -DC frequency=0 Hz

**Depending upon Phase angle θ, elements are**

| θ | Elements |
|---|---|
| $0^o$ | R |
| $90^o$ | $L_1$ |
| $-90^o$ | $C_1$ |
| $0^o < \theta_1 < 90^o$ | $R_1$, $L_1$ |
| $-90^o < \theta_1 < 0^o$ | $R_1$, $C_1$ |

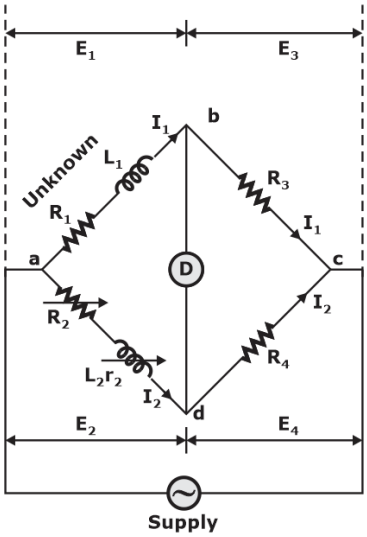
**Convergence to balance point:**

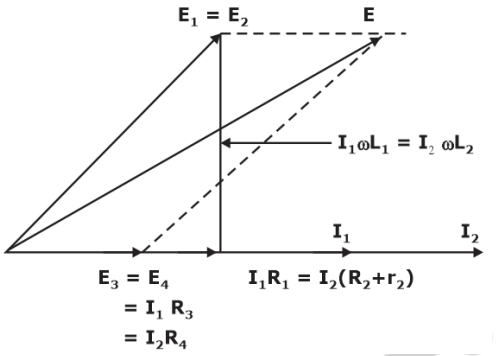
If the variables are in the same arm of the bridge then minimum time is required for balancing of bridge. This is called convergence to balance point.

Quality factor (Q.F.), $Q.F. = \dfrac{\text{Energy Stored}}{\text{Energy disspiated}}$

**Measurement of Self Inductance**

1. Maxwell's Inductance Bridge




$$R_1 = \frac{R_3}{R_4}(R_2 + r_2), \quad L_1 = \frac{R_3}{R_4} \cdot L_4$$

Where,

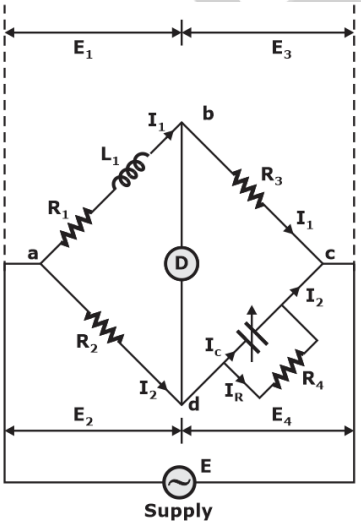
$L_1$ = Unknown inductance of resistance $R_1$

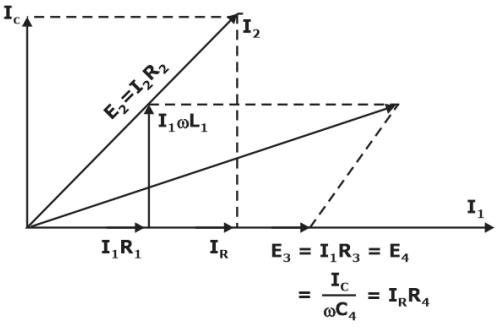
$L_2$ = Variable inductance of fixe resistance $r_2$

$R_2$ = Variable resistance connected in series with $L_2$

$R_3$, $R_4$ = Known non-inductive resistance

**2. Maxwell's Inductance-capacitance Bridge**



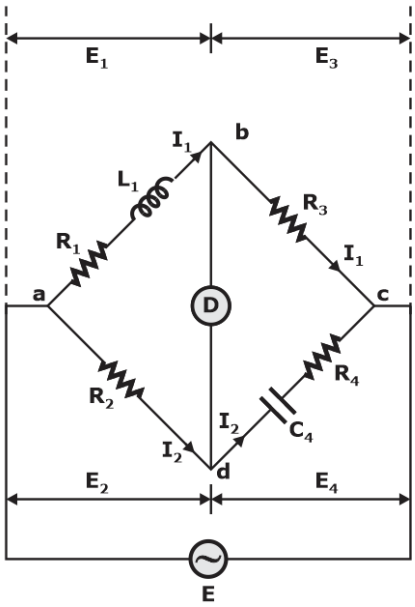

$$R_1 = \frac{R_2 R_3}{R_4} \text{ and } L_1 = R_2\, R_3\, C_4$$

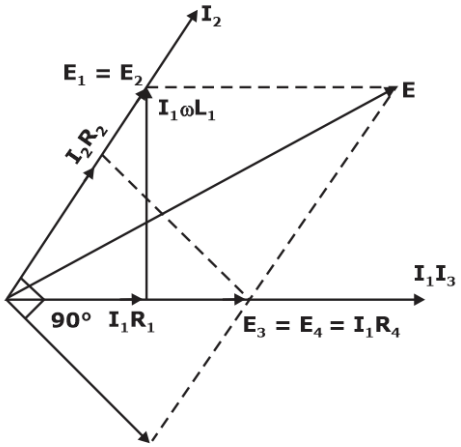
Q factor of the coil, $Q = \frac{\omega L_1}{R_1} = \omega C_4 R_4$

**Note:**

- Not suitable for measurement of high Q coil because phase angle criteria do not satisfy.
- Not suitable for measurement of low Q-coil because of sliding balance problem.
- Suitable for measurement of medium Q coil, i.e. (1 < Q < 10).

### 3. Hay's Bridge




$$R_1 = \frac{\omega^2 R_2 R_3 R_4 C^2_4}{1+\omega^2 C^2_4 R^2_4} = \frac{R_2 R_3}{R_4}\left(\frac{1}{1+Q^2}\right)$$

$$L_1 = \frac{R_2 R_3 C_4}{1+\omega^2 C^2_4 R^2_4} = \frac{R_2 R_3 C_4}{1+\left(\frac{1}{Q}\right)^2}$$

For Q > 10

$$L_1 = R_2 R_3 C_4$$

$$Q = \frac{\omega L_1}{R_1} = \frac{1}{\omega C_4 R_4}$$

Where,

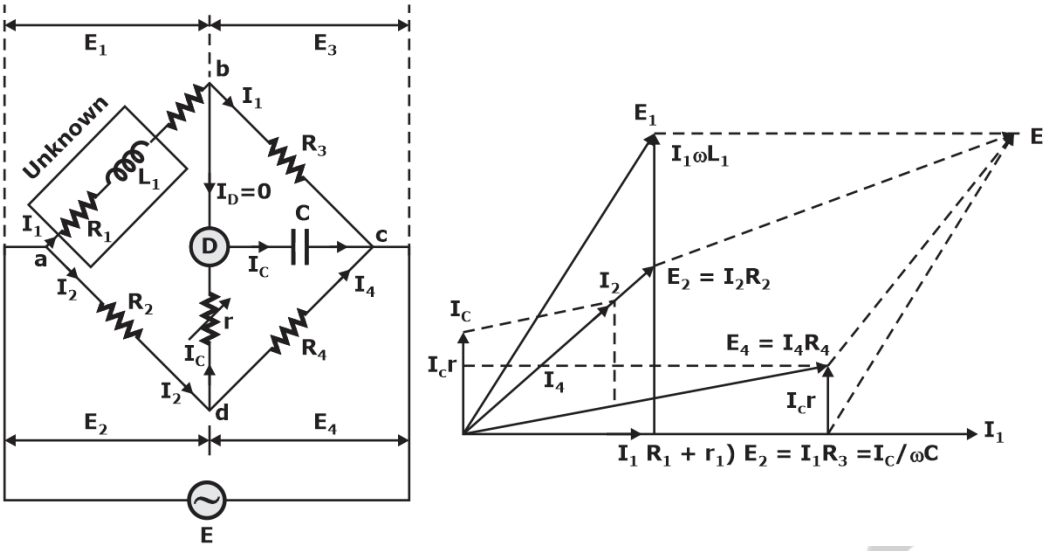
$L_1$ = Unknown-inductance having a resistance R1

$R_2$, $R_3$, $R_4$ = Know non-inductive resistance

$C_4$ = Standard capacitor

**Note:** The Hay's bridge is suited for measuring high Q inductors.

### 4. Anderson's Bridge




$$R_1 = \frac{R_2R_3}{R_4} - r_1$$

$$L_1 = \frac{CR_3}{R_4}\left[r\left(R_4 + R_2\right) + R_2R_4\right]$$

Where,

$L_1$ = Self-inductance to be measured

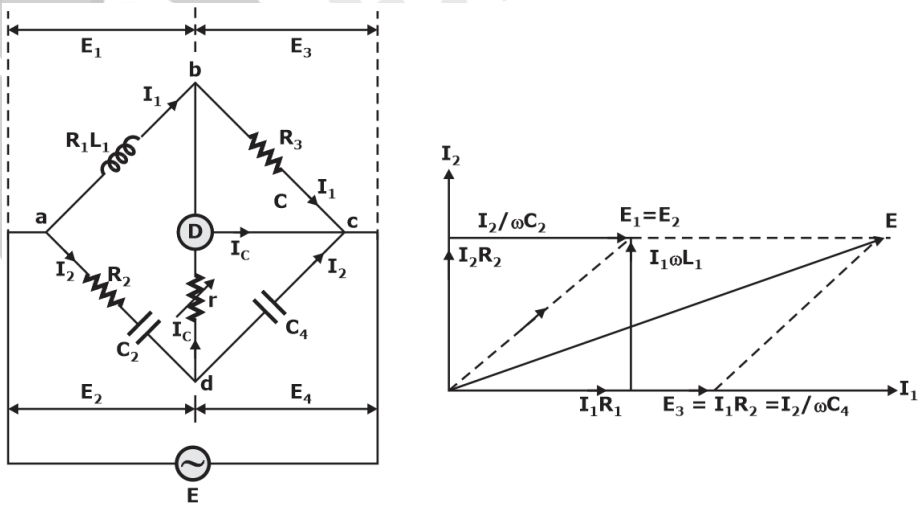
$R_1$ = Resistance of self-inductor

$r_1$ = Resistance connected in series with self-inductor

r, $R_2$, $R_3$, $R_4$ = Known non-inductive resistances

C = Fixed standard capacitor

## 5. Owen's Bridge




$$L_1 = R_2\, R_3\, C_4$$

$$R_1 = R_3\,\frac{C_4}{C_2}$$

Where,

$L_1$ = Unknown self-inductance of resistance $R_1$

$R_2$ = Variable non-inductive resistance

$R_3$ = Fixed non-inductive resistance

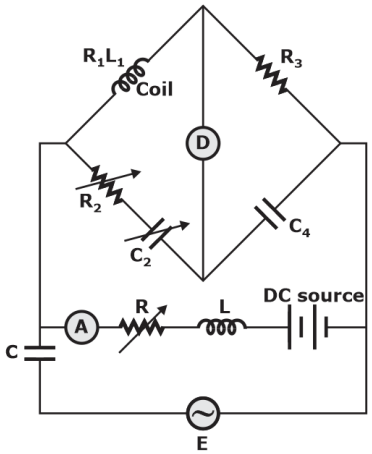
$C_2$ = Variable standard capacitor

$C_4$ = Fixed standard capacitor

**Note:**

Owen's bridge is used for the measurement of unknown inductance and incremental inductance and incremental permeability (μ).

**Measurement of Incremental Inductance**




Incremental inductance, L1 = $R_2$ $R_3$ $C_4$

Incremental permeability, $\mu = \frac{L_1 i}{N^2 A}$

Where,

N = Number of turns

A = Area of flux path

l = Length of flux path

$R_2$ = Variable non-inductive resistance

$R_3$ = Fixed non-inductive resistance

$C_4$ = Fixed standard capacitor

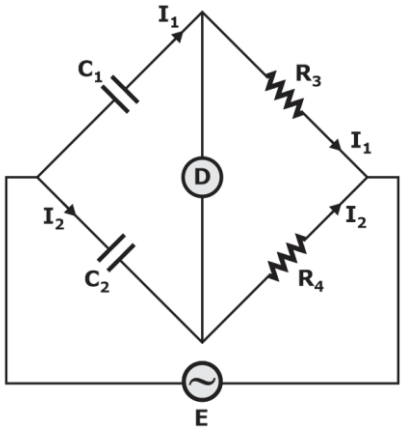
**Note:**

- External D.C. Source is used to compensate residual magnetism.
- Capacitor, C is to block D.C. from entering into A.C. and inductor, L is to block A.C. from entering into D.C.

**Measurement of Capacitance**

**1. De Sauty's Bridge**

**(a) For lossless capacitor**




$$C1 = \frac{C_2R_4}{R_4}$$

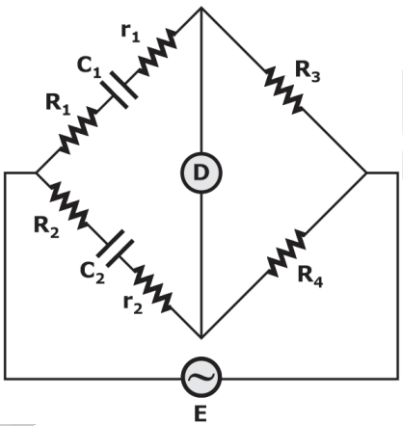
Where,

$C_1$ = Capacitor whose Capacitance to be measured

$C_2$ = A Standard capacitor

$R_3$, $R_4$ = Non-inductive resistors

(b) For imperfect capacitor having dielectric loss




$$\frac{C_1}{C_2} = \frac{R_4}{R_3} = \frac{R_2 + r_2}{R_1 + r_1}$$

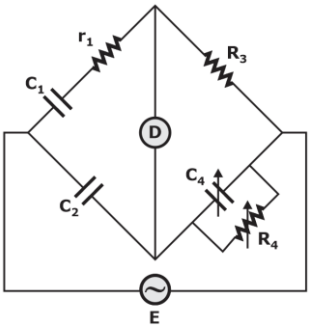
Where,

$r_1$, $r_2$ = Resistance representing the loss component of the two capacitors.

**Dissipation factor**

$D = \omega C_1 r_1 = \omega C_2 r_2$

**2. Schering Bridge**




$$r_1 = \frac{R_3 C_4}{C_2}$$

$$\mathbf{C_1 = r_1 C_2 \left(\frac{R_4}{R_2}\right)}$$

**Dissipation factor**

$D = \omega C_1 r_1 = \omega C_2 R_2$

Where,

$C_1$ = capacitor whose capacitance is to be determined

$r_1$ = Series resistance representing the loss in the capacitor $C_1$

$C_2$ = Standard loss-free capacitor

$R_3$ = Non-inductive resistance

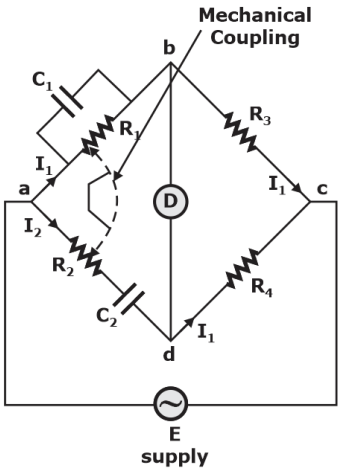
$C_4$ = Variable capacitor

$R_4$ = Variable non-inductive resistance in parallel with variable capacitor $C_4$

**Note:** Schering bridge is shielded with a metal screen to reduce the stray capacitance exists between the arms and arms to the earth.

**Measurement of Frequency**

**Wien's Bridge**




**Frequency**

$$\frac{R_4}{R_3} = \frac{R_2}{R_1} + \frac{C_1}{C_2}$$

$$f = \frac{1}{2\pi\sqrt{R_1R_2C_1C_2}}\ Hz$$

For $R_1 = R_2 = R$ and $C_1 = C_2 = C$

$$f = \frac{1}{2\pi RC}$$

$$\frac{R_4}{R_3} = 2$$

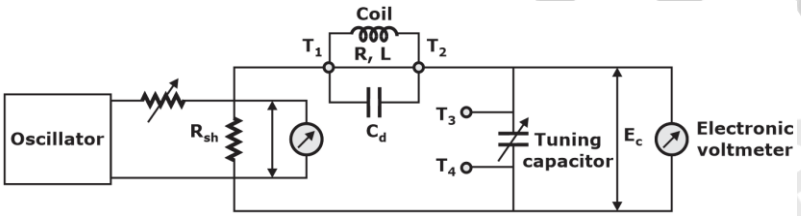
**Limitation of Wein's Bridge**

If the input signal is not a sinusoidal or a signal containing harmonics, then bridging is not possible because the null detector is sensitive to the frequencies.

**Q-meter**

It works on the principle of series resonance.

**Measurement of the Storage Factor Q**




The resonant frequency of Q-Meter, $f_0 = \frac{1}{2\pi\sqrt{LC}}$

Measured value of Q, $Q_m = \frac{\omega_0 L}{R + R_{sh}}$

True value of Q, $Q_t = \frac{\omega_0 L}{R} = Q_m = \left(1 + \frac{R_{sh}}{R}\right) = Q_m\left(1 + \frac{C_d}{C}\right)$

Where, R = Resistance of coil, L = Inductance of coil, $R_{sh}$ = Shunt resistance
C = Tuning Capacitance, $C_d$ = Distributed or self–capacitance

Measurement of inductance, $L = \frac{1}{4\pi^2 f_0^2 C}$

Measurement of effective resistance, $R = \frac{\omega_0 L}{Q_t}$

Measurement of Distributed or self–capacitance

Resonance frequency $f_1 = \frac{1}{2\pi\sqrt{L(C_1 + C_d)}}$ , $f_2 = \frac{1}{2\pi\sqrt{L(C_2 + C_d)}}$

When, $f_2 = nf_1$ then, $C_d = \frac{C_1 - n^2C_2}{n^2 - 1}$

where, $C_1$ = Tuning capacitance at frequency $f_1$, $C_2$ = Tuning capacitance at frequency $f_2$

Measurement of Unknown Capacitance $C_x$

Adjust capacitor $C = C_1$ to get resonance frequency f1 with unknown capacitance $C_x$ in parallel.

$$f_1 = \frac{1}{2\pi\sqrt{L(C_x + C_1)}} \qquad \text{...(i)}$$

Now remove $C_x$ and again adjust $C = C_2$ to get same resonance frequency $f_1$

$$f_1 = \frac{1}{2\pi\sqrt{LC_2}} \quad \ldots(ii)$$

By equating equation (i) and (ii)

$C_x = C_2 - C_1$

# Chapter-5

# Digital Instruments & Instrument Transformers

## Cathode Ray Oscilloscope

CRO is a digital instrument, which works on the principle of thermionic emission i.e. emission of electron from a heated surface. It is a linear device. With the use of CRO, one can measure peak to peak, rms, peak or average value of voltage and current.

## Calibration of CRO

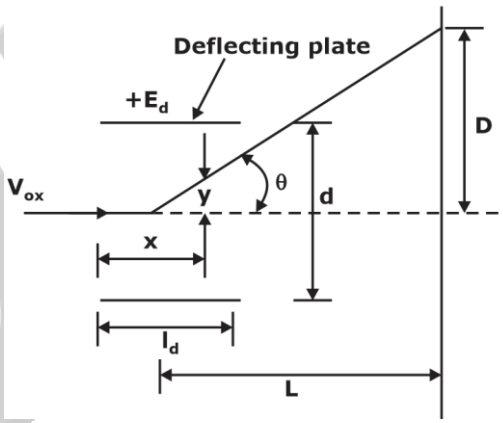
Calibration of CRO is done by applying a known quality of square signal.

The rise time ($t_r$), of signal applied to CRO and bandwidth of CRO are related as

$t_r \times B.W. = 0.35$

If this condition fails, then the signal is distorted at the output of CRO.

## Electrostatic Deflection




$$y = \frac{1}{2}\frac{eE_y}{mv_{ox}^2}x^2$$

where, y = Displacement in y-direction; m, e = Charge of an electron; Coulomb

$E_y$ = Electric field intensity in Y-direction; V/m, m = Mass of electron; kg

$V_{ox}$ = Velocity of electron when entering the fields of deflecting plates; m/s

x = Displacement in x-direction; m

**Deflection**

$$D = \frac{L\, l_d\, E_d}{2dE_a}$$

Where,

L = Distance between screen and the center of deflecting plates; m

$l_d$ = Length of deflecting plates; m,

$E_d$ = Potential between deflecting plates; V

d = Distance between deflecting plates; m,

$E_a$ = Voltage of pre-accelerating anode; V

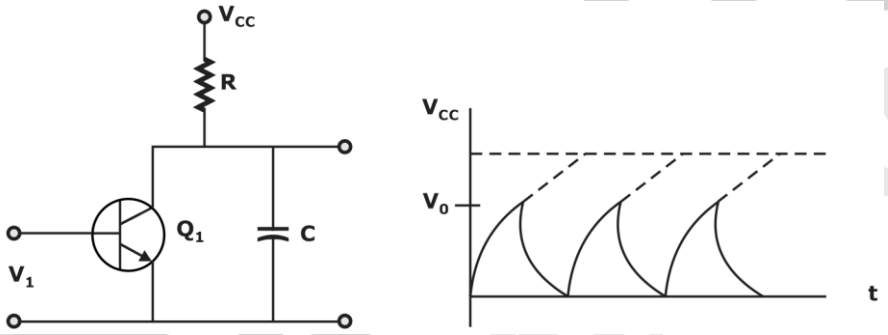
**Deflection sensitivity:**

$$S = \frac{D}{E_d} = \frac{L\, l_d}{2dE_a} \quad m/v$$

**Deflection factor:**

$$G = \frac{1}{S} = \frac{2dE_a}{L\, l_d} \quad v/m$$

**Sawtooth Generator:**




$V_0 = V_{cc}\,[1 - \exp(-t/RC)]$

Where,

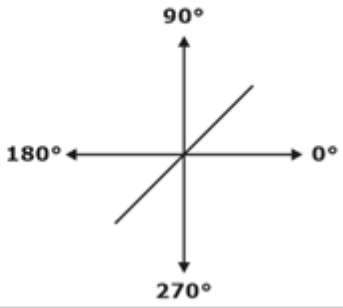
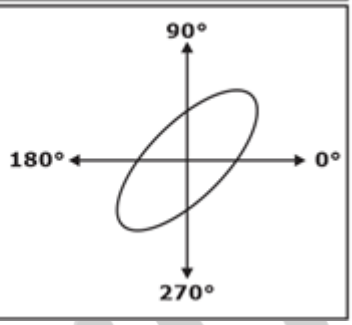
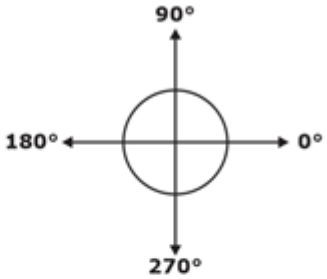
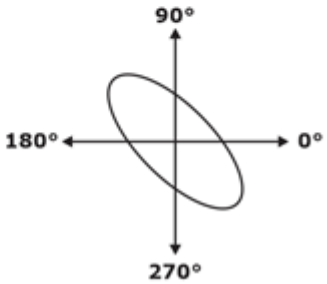
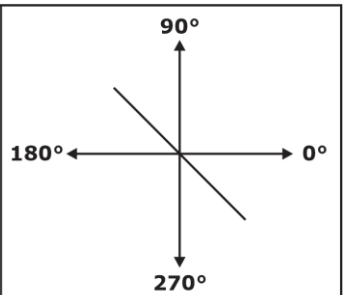
$V_0$ = Instantaneous voltage across the capacitor at time t; $V_{cc}$ = Supply voltage

**Lissajous patterns**

If horizontal and vertical deflecting plates are applied with sinusoidal signal, the waveform pattern appearing on the screen is called Lissajous pattern.

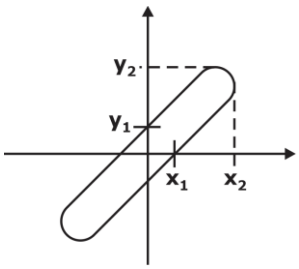
**Application**

- Used to find the phase angle difference between the signal applied to vertical and horizontal plates.
- Used for finding the frequency ratio between vertical and horizontal plates voltage.

| Phase angle (f), between $V_a$ and $V_y$ | Lissajous pattern |
|---|---|
| 0° or 360° |   |
| $0° < \phi < 90°$ or $270° < \phi < 360°$ |   |
| $\phi = 90°$ or $270°$ |   |
| $90° < \phi < 180°$ or $180° < \phi < 270°$ |   |
| $\phi = 180°$ |   |

**Finding the phase angle ϕ from given Lissajous pattern**

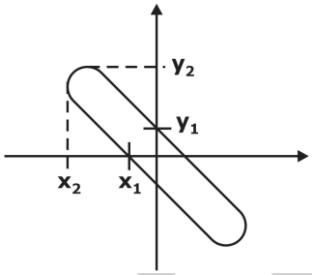
(a) When Lissajous pattern is in first and third quadrants




First possibility, $\phi = \sin^{-1}\left(\frac{x_1}{x_2}\right) = \sin^{-1}\left(\frac{y_1}{y_2}\right)$

Second possibility = 360° – ϕ

(b) When Lissajous pattern is in second and fourth quadrant




First possibility

$\phi = 180^{o} - \sin^{-1}\left(\frac{x_1}{x_2}\right)$

Second possibility = 360° – ϕ

**Measurement of Frequency Using Lissajous Pattern**

$$\frac{f_y}{f_x} = \frac{\text{(number of intersections of the horizontal line with the curve)}}{\text{(number of intersections of the vertical line with the curve)}}$$

Where,

$f_y$ = Frequency of signal applied to Y plates

$f_x$ = Frequency of signal applied to X plates

**Digital Meters**

Basic measurable quantity in digital meter is D.C.

1. **Resolution (R) of Digital Meter**

The smallest change in the input that a digital meter can detect is called resolution.

$R = \frac{1}{10^n}$

Where, n = Number of full digit.

2. **Sensitivity (S)**

The smallest change in input that can be displayed within given range.

S = Resolution × Range of meter

3. **Over ranging**

   Switch on the extra half (1/2) is called over ranging. Due to this over ranging, the range of the instrument increases.

## Instrument Transformers

### Ratios of Instrument Transformers:

1. **Transformation Ratio (R)**

   It is the ratio of the magnitude of the primary phasor to the secondary phasor.

$$R = \frac{|\text{primary phasor}|}{|\text{secondary phasor}|}$$

For current transformer (C.T.), $R = \frac{\text{primary winding current}}{\text{secondary winding current}}$

For potential transformer (P.T.), $R = \frac{\text{primary winding voltage}}{\text{secondary winding voltage}}$

2. **Nominal Ratio ($K_n$)**

   It is the ratio of rated primary winding current (or voltage) to the rated secondary winding current (or voltage)

For C.T., $K_n = \frac{\text{rated primary winding current}}{\text{rated secondary winding current}}$

For P.T., $K_n = \frac{\text{rated primary winding voltage}}{\text{rated secondary winding voltage}}$

3. **Turns Ratio (n)**

For C.T., $n = \frac{\text{number of turns of secondary winding}}{\text{number of turns of primary winding}}$

For P.T, $n = \frac{\text{number of turns of primary winding}}{\text{number of turns of secondarywinding}}$

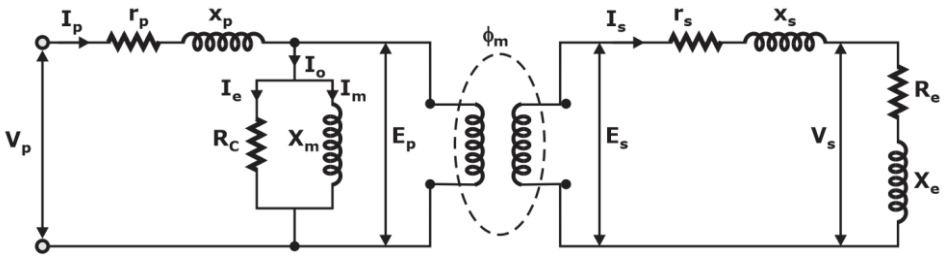
4. **Ratio Correction Factor**:

$$RCF = \frac{R}{K_n}$$

**Remember:**

The ratio marked on the transformers is their nominal ratio.

### Current Transformer Equivalent Circuit




Where,

$r_s$, $X_s$ = resistance, reactance of secondary winding

$r_e$, $X_e$ = resistance , reactance of external burden

$E_p$, $E_s$ = primary and secondary winding induced voltage

$N_p$, $N_s$ = number of primary and secondary winding turns

$I_p$, $I_s$ = primary and secondary winding current

ϕ= flux in transformer

θ = phase angle of transformer

δ = angle between secondary winding induced voltage and secondary winding current

Δ = phase angle of secondary winding load circuit

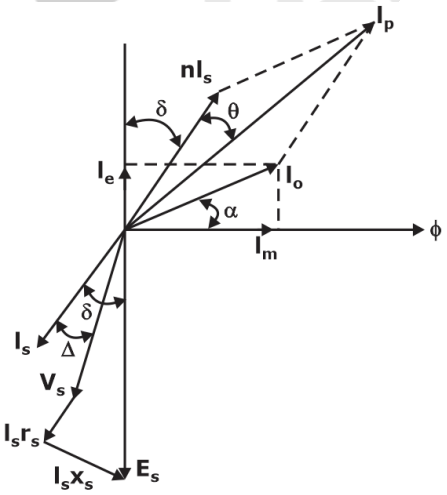
$I_o$ = exciting current

$I_m$ = magnetizing component of exciting current

$I_e$ = loss component of exciting current

α = angle between exciting current and flux

**Phasor Diagram**




**Transformation ratio**

$$R = \frac{I_P}{I_S} = \frac{nI_S + I_0 \cdot \sin(\delta + \alpha)}{I_S} = n + \frac{I_0}{I_S}\sin(\delta + \alpha)$$

$$R = n + \frac{I_0}{I_S}\sin(\delta + \alpha) = n + \frac{I_0}{I_S}(\sin\delta\cos\alpha + \cos\delta\sin\alpha)$$

$$R = n + \frac{I_M \sin\delta + I_C \cos\delta}{I_S}$$

Where, $I_m = I_o \cos\alpha$

$I_C = I_o \sin\alpha$

**Phase angle**

For very small angles, $\theta \approx \frac{I_0 \cdot \cos(\delta + \alpha)}{nI_S + I_0 \cdot \sin(\delta + \alpha)}$

This expression can still be simplified with the assumption $I_0 << nI_s$.

$$\theta \approx \frac{I_0 \cdot \cos(\delta + \alpha)}{nI_S} \text{rad}$$

$$\theta \approx \frac{I_0(\cos\delta\cos\alpha - \sin\delta\sin\alpha)}{nI_S} \approx \frac{I_M \cos\sigma - I_C \sin\delta}{nI_S} \text{rad}$$

Phase angle of CT is given by

$$\theta \approx \frac{180}{\pi}\left(\frac{I_M \cos\delta - I_C \sin\delta}{nI_S}\right) \text{degrees}$$

**Ratio error**

$$\text{Ratio error} = \frac{\text{nominal ratio } (K_n) - \text{actual ratio } (R)}{\text{actual ratio } (R)}$$

**Remember:**

- The primary current of C.T depends on load connected to the system but does not depend on the secondary winding burden.
- Primary winding is single turn or bar winding and secondary has more number of turns to reduce the current in secondary.
- If primary current is very high, it causes reduction is ratio error and phase angle error. So to increase value of primary current the primary is maintained with single turn.
- The secondary number of turns is reduced by 1 or 2, then the ratio error reduces.

**Potential transformer**

Actual transformation (voltage) ratio

$$R = n + \frac{\frac{I_s}{n}\left[R_P \cos\Delta + X_p \sin\Delta\right] + I_e r_p + I_m x_P}{V_s}$$

Phase angle

$$\theta = \frac{I_s}{V_s}\left(X_s \cos\Delta - R_s \sin\Delta\right) + \frac{I_e X_p - I_m r_p}{nV_s} \text{rad.}$$

**Note:**

- C.T never operates with secondary winding open but P.T can be operated with secondary winding open.

- Strip wound core is used to reduce ratio and phase angle errors.

**Application of C.T and P.T**

- Multiple operation with a single device.
- High current and higher voltage are step down to lower current and lower voltage so that metering is easier.
- Measuring circuit is isolated from the power circuit.
- Low power consumption.
- Replacement is easier.

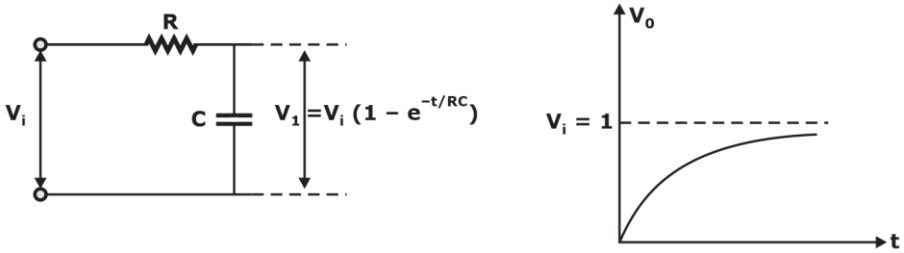
## Potentiometer

1. **Zero Order System**

   As input changes, output also changes immediately called zero order system. Examples: Resistor.

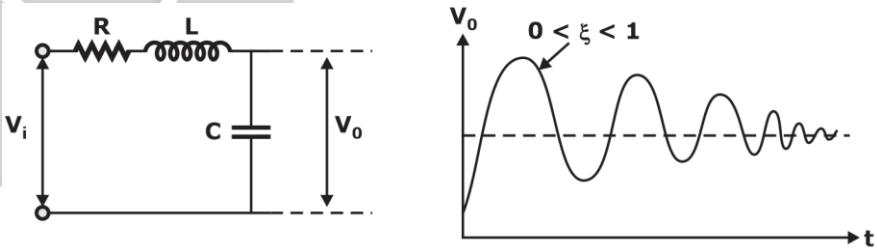
2. **First Order System**

   As input changes, output also changes but not immediately, it takes some delay but without oscillation. Example: heater.




3. **Second Order system**

   As input changes, output also changes, with some delay and oscillation.




**Remember:**

The analog instruments are of second order instrument which has damping factor (ξ) between 0.6 to 0.8. It is under damped.

### Standards

| Quantity | Limit | Definition |
|---|---|---|
| Length | Meter | The length of path travelled by light in an interval of $\frac{1}{299792458}$ sec. |
| Time | Second | 9.192631770 × 109 cycles of radiation from vaporized ceslum-133 atom. |
| Temp. | Kelvin | The temperature difference between the absolute and the triple point of water is defined as 273.16°K. |
| Voltage | Volt | Standard cell voltage of Weston cell i.e., 1.0186 V. |
| Current | Ampere | One ampere is the current flowing through two infinite cross section placed 1 meter apart produced a force of 2 ×10−7 N/m. |

## Magnetic Measurements

### Flux density

$B = \frac{\phi}{A_s} = \frac{RK_q\theta_1}{2NA_s}$

Where, ϕ = Flux linking search coil, $A_s$ = Cross-sectional area of specimen,

R = Resistance of the ballistic galvanometer circuit

$K_q\theta_1$ = Charge indicated by ballistic galvanometer

N = Number of turns in the search coil Hysteresis loss per unit volume, $P_h = \eta f B_m k$

Where, η = Hysteresis coefficient, f = Frequency; Hz, $B_m$ = Maximum flux density; Wb/m2

k = Steinmetz coefficient

**Note:** The value of k varies from 1.6 to 2.

Eddy current loss per unit volume for laminations, $Pe = \frac{4k_f^2 f^2 B_m^2 t^2}{2\rho}$

Where, $k_f$ = Form factor, t = Thickness of laminations; m, ρ = Resistivity of material; Ω-m

Total iron loss per unit volume, $p_1 = p_h + p_e = \eta f B_m k + \frac{4k_f^2 f^2 B_m^2 t^2}{3\rho}$

Maximum flux density, $B_m = \frac{E_2}{4k_f f A_s N_2}$

For sinusoidal supply, $B_m = \frac{E_2}{4.44 f A_s N_2}$

Where, $E_2$ = Voltage induced in secondary winding, $E_2 = 4k_f f \phi_m N_2$

$K_f$ = Form factor (= 1.11 for sinusoidal), f = Frequency

$A_s$ = Cross-sectional area of specimen, $N_2$ = Number of turns in secondary winding

$\phi_m$ = Maximum flux linkage

****