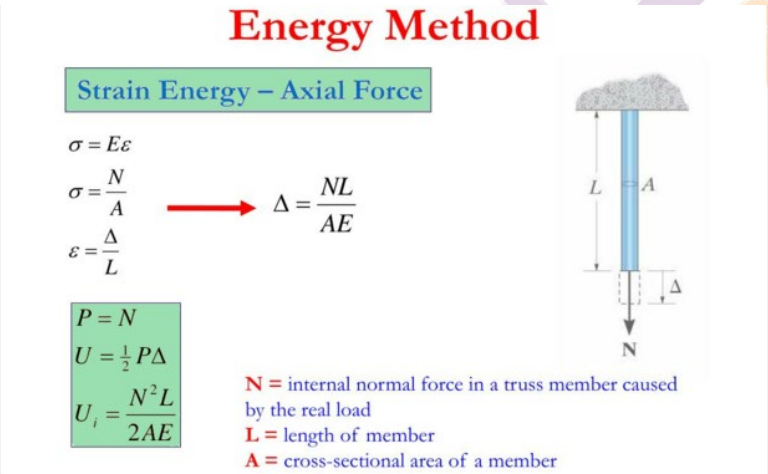
Strain Energy Method

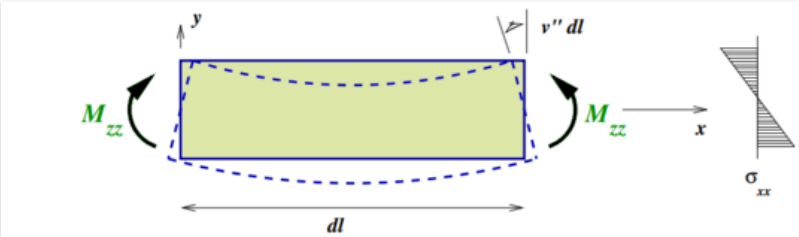
Strain energy is calculated by the work done by the structure's member to deflect the member under the action of external loads. Hence, this strain energy can be used to calculate the deflection at any point in the member caused due to external loads. This method of calculating the deformation is known as the strain energy method.

Strain energy can be recovered in the member up to the elastic limit of stresses in the structure. The area under the stress-strain curve up to the elastic limit is the strain energy per unit volume.



Strain Energy in Beams

Strain energy in beams can be used to calculate the deformation of the beam due to the external loads. In the case of beams, transverse loads cause a bending moment, which introduces the bending stresses. In beams, this strain energy caused by moment can be used to find the deflection at any point, and this method is known as the strain energy method.



As in the axially loaded bar, $\sigma_{yy} = 0$, $\sigma_{zz} = 0$, $\tau_{xy} = 0$, $\tau_{xz} = 0$, and $\tau_{yz} = 0$, and

$$U = \frac{1}{2}\int_V \sigma_{xx}\epsilon_{xx}\, dV \ .$$

For bending, $\sigma_{xx} = My/I$ and $\epsilon_{xx} = My/EI$. Substituting $dV = dA\, dx$,

$$U = \frac{1}{2}\int_L \int_A \frac{M(x)^2\, y^2}{E(x)\, I(x)^2}\, dA\, dx \ ,$$

where $\int_A y^2 dA = I$, so

$$U = \frac{1}{2}\int_L \frac{M(x)^2}{E(x)\, I(x)}\, dx \ .$$

Strain Energy Method for Indeterminate Structures

In the case of indeterminate structures for calculating the deflection, mainly displacement methods are used because deformation and rotations are unknown in the displacement method. The displacement method can be preferred over the force method when the number of unknown forces exceeds the number of unknown displacements.

The strain energy method can also calculate deflections in indeterminate structures. This method is based on the Castigliano theorem for deflection. Castigliano's first theorem is used as the displacement method, while Castigliano's second theorem is used as the force method for calculating the deflections of indeterminate structures.

Strain Energy Method for Finding Deflection

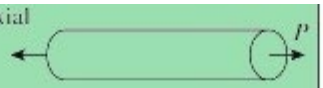
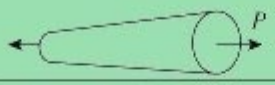
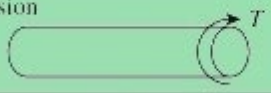
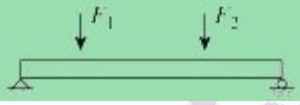
The strain energy method for finding deflection is based on the Castigliano theorems. This theorem relates the strain energy of the structure with the external point load at the displacement location. There are two Castigliano's theorems in the structural analysis related to the displacement of the structure.

Castigliano's first theorem: According to the first theorem, the first partial derivative of the total strain energy of structure concerning any deflection component is equal to the point load acting at that point in the same direction as that of deflection.

Castigliano's second theorem: According to the second theorem, the first derivative of total strain energy concerning the point load acting at that point is equal to the deflection at the same point of structure in the same direction of deflection

Problems on Strain Energy Method

The strain energy method is a method to calculate deflection at any point in the structure. This method uses the concept of total strain energy in the structure. Here, the formula of strain energy for some standard cases of beams is given, which helps solve the problems of the strain energy method.

Prismatic bar with constant force	Axial 	$U = \frac{L}{2EA} P^2$
Nonprismatic bar (general)		$U = \int_0^L \frac{P^2}{2EA} dx^2$
Beam bending (general)	Moment, vertical force 	$U = \int_0^L \frac{M^2}{2EI} dx$
Torsional shear (circular)	Torsion 	$U = \frac{L}{2GJ} T^2$
Transverse shear (general)	Moment, vertical force 	$U = \int_0^L \frac{CV^2}{2AG} dx$ C = 1.20 (rectangular), 1.11 (circular), 2.00 (thin wall, round)

Different Methods for Finding Deflection

As we know, the calculation of the deflection of a member is a major part of the analysis of a structure. So, there are various methods available for calculating deflections. Here are some methods are listed below, which can be used to calculate the deflection of beams. The suitability of these methods depends upon the type of loading and support conditions etc.

- Double Integration method
- Macaulay's method (It is the extension of the double integration method)
- Moment-area method
- Strain energy method (based on Castigliano's method)
- Superposition method
- Conjugate beam method