Castigliano’s Theorem

Castigliano’s structural analysis theorem relates a structural member's deformation to the strain energy available within the structure. This theorem provides a method of determining the displacements of a linear elastic body based on the partial derivative of the strain energy.

Strain Energy: It is defined as the energy stored in a structural member due to its deformations. It can also be calculated as the area under the stress-strain curve upto the elastic limit. The area under the stress-strain curve gives the strain energy per unit volume or strain energy density. Castigliano’s theorems are based on the partial derivatives of the strain energy.

Castigliano also gives a theorem related to the theorem of least work; according to that, redundant reaction forces in a statically indeterminate structure will be such that it makes the strain energy to its minimum value. Castigliano's theorem for deflection and slope are the same theorems, but with the help of these theorems, a relation between slope and deflection of a member can be established. There are two Castigliano’s theorems available, which are used in structural analysis.

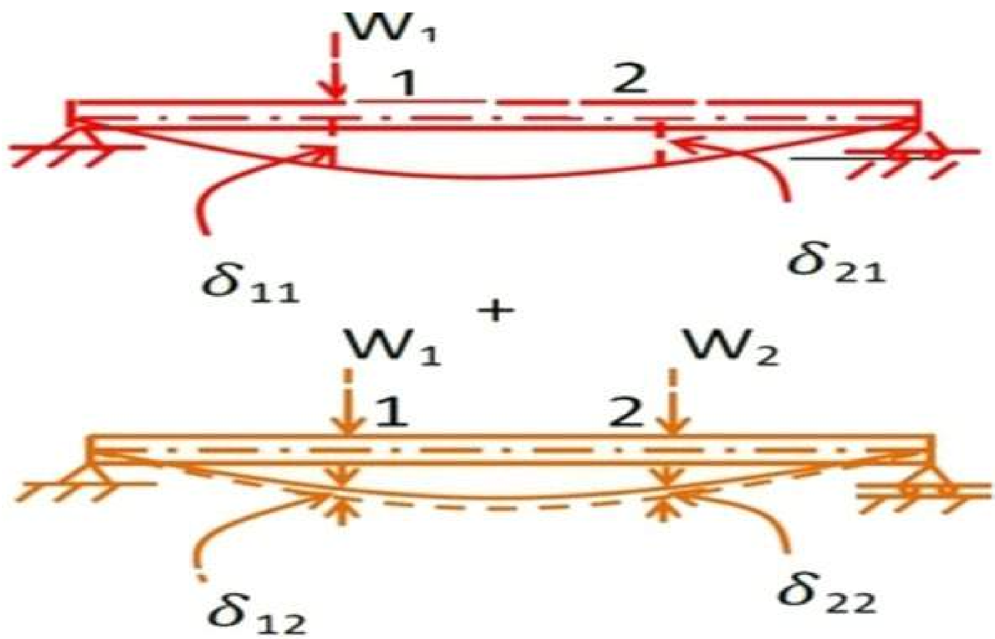
$$\Delta = \int_0^L \frac{M}{EI}\left(\frac{\partial M}{\partial P}\right) dx$$

$$\theta = \int_0^L \frac{M}{EI}\left(\frac{\partial M}{\partial M'}\right) dx$$

Castigliano's Theorem Derivation

As we know, with the help of castigliano’s theorem, Many complex deflection problems can be solved easily. So, we should understand the basics of these theorems. Here, we will derive the first theorem, and the second theorem can be understood similarly.

Castigliano's first theorem derivation:



$\delta_{22}=\Delta_{22}W_2$; With Δ_{22}= deflection at 2 due to unit load at 2 and,

$\delta_{12}=\Delta_{12}W_2$; With Δ_{12}= deflection at 1 due to unit load at 2,

then $\delta_1=\delta_{11}+\delta_{12}=\Delta_{11}W_1+\Delta_{12}W_2$ I

similarly, $\delta_2=\delta_{21}+\delta_{22}=\Delta_{21}W_1+\Delta_{22}W_2$ II

Considering the work done=U_i

$$U_i= 1/2[W_1\delta_{11}+W_2\delta_{22}] + W_1\delta_{12}$$

$$U_i= 1/2[\Delta_{11}W_1^2+\Delta_{22}W_2^2] + \Delta_{12}W_1W_2 \text{ ...III}$$

From equation III,

$$U_i= 1/2[\Delta_{11}W_1^2+\Delta_{22}W_2^2] + \Delta_{12}W_1W_2$$

$$\delta U_i/\delta W_i = \Delta_{11}W_1 + \Delta_{12}W_2 =\delta_1$$

This is Castigliano's first theorem.

Castigliano's First Theorem

Castigliano’s theorem relates the strain energy with the point load acting on the member. According to Castigliano’s first theorem, the First partial derivative of total strain energy available in the structure with respect to the deflection component at any point is equal to the point load acting at the deflection point in the same direction.

Castigliano's first theorem is applicable to the linearly or nonlinearly varying elastic structures with the condition of constant temperature and unyielding of the supports. Castigliano’s first theorem is used to find the value of load causing the deflections.

Castigliano's Second Theorem

Castigliano’s theorem also relates to the strain energy with the deflections of the member in its second theorem. According to Castigliano’s second theorem, the first partial derivative of the total strain energy with respect to the point load acting on the member is equal to the deflection of the member at the point of loading in the same direction of load.

$$\delta = \partial U / \partial P$$

Castigliano’s second theorem applies to the elastic body within its limit. Both Castigliano’s theorem is applicable up to the deformations and force relation are linear.

Uses of Castigliano’s theorem

Castigliano's theorem is used for calculating the deflections under the action of point load with the help of strain energy. These theorems are based on the principle of the virtual work energy theorem, according to which work done by the external forces in the structure is equal to the work done by the set of internal forces within the structure. These theorems can also be used to calculate deflections in the case of non-prismatic sections.

The strain energy method for calculating the deflections in the beams and trusses is based on Castigliano’s second theorem. With this method's help, many structural analysis problems can be solved. And with the help of Castigliano’s first theorem, the load can be found in the case of deflection of trusses, so it has many advantages in the case of truss analysis.

Castigliano's Theorem Applications

Castigliano’s theorem has many applications in structural analysis; It is used to find out deflections in beams, trusses, and frames. Castigliano’s method is very useful for efficiently solving deflection problems in statically indeterminate beams with many kinematical constraints. With the help of considering strain energy due to shear stress, the shear effect can be considered in calculating the deflections on structures.

Castigliano’s theorem has many applications in solving the problems for deflections in the case of structures having varying cross-sectional areas. With the help of these theorems, deflection problems can be solved easily. Deflection problems are very complex in the case of frames and trusses, but with the help of Castigliano’s theorems, they can be solved easily.

Castigliano's Theorem Limitations

Castigliano’s theorem is very useful for calculating the deflections of beams, trusses, frames, and other structures. These theorems are useful because of some assumptions made for the analysis of structures. After violating these assumptions, theorems can’t be used, and such conditions become limitations of the theorem.

As in Castigliano’s method, one assumption has been made that the distribution of elastic forces is the same as that of external forces in the structure. And this assumption violates the principle of continuity of deformation. So it became one of the limitations of this theorem.

Castigliano's Theorem Examples

As we discussed, Castigliano’s theorem is very useful for solving deflection problems in structural analysis. So it's important to understand in a better way. Here a few examples are provided which make the concepts strong:

Example: Consider a cylinder attached to the fixed wall with a diameter of 4 cm and a length of 2 m. Find the displacement of a cylinder when it is subjected to a torque of 8 N.m. (Assume G= 120 GPa).

Solution

Formula used $\delta = \frac{\delta}{\delta T}\left[\int_0^L \frac{T^2 dx}{2GJ}\right]$ and $J = \frac{\Pi}{32}d^4$

As, Area and moment of inertia are not changing,

So, it can be converted into; $\delta = \frac{\delta}{\delta T}\left[\int_0^L \frac{T^2 dx}{2GJ}\right] = \int_0^L \frac{\delta}{\delta T}\left[\frac{T^2}{2GJ}\right]dx = \int_0^L\left[\frac{Tdx}{GJ}\right] = \frac{TL}{GJ}$

$$\delta = \frac{8\times 2}{120\times 10^9\times\frac{\Pi}{32}\times(0.04)^4} = 0.5305\ mm$$