IMPORTANT FORMULAS ON STRUCTURAL ANALYSIS

1. STATIC INDETERMINACY

1.1. External Indeterminacy

Mathematically, external indeterminacy can be expressed as follows.

$$\boxed{D_{Sc} = r - s}$$

Where,

r = total number of unknown support reactions.

S = total number of equilibrium equations available.

External indeterminacy of various cases

1.2. Internal Indeterminacy

Case 1: Beam

There is no internal indeterminacy for beams because if we know the support reactions, we can find the axial force, shear force and bending moment at any section in the beam.

Case 2: Trusses

The internal indeterminacy for the trusses can be determined by following expression.

$$D_{Si} = m - (2j - 3); \text{ for plane truss}$$

$$D_{Si} = m - (3j - 6); \text{ for space truss}$$

Where,

m = number of members

j = number of joints

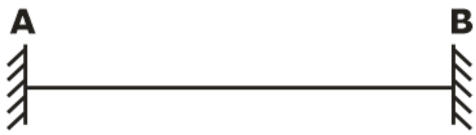
2. KINEMATIC INDETERMINACY (D_K)

It is defined as the number of independent displacements at all joints in a structure. Displacement are counted always only at the joints. Displacement includes slopes and deflection. Wherever the cross-section area, changes or material changes then it is treated as a joint in any structure.

The kinematic indeterminacy can be determined for various cases as follows.

Case 1: Beams

Example:



→ Displacement at A and B in x-direction is zero

→ Displacement at A and B y-direction is zero

→ Rotation at A and B is zero

∴ Degree of freedom = $D_k = 0$

D_k(inextensible) = D_k (extensible) – Number of independent displacement prevent.

Note: It net given in the question, then assume that members are extensible.

Example:

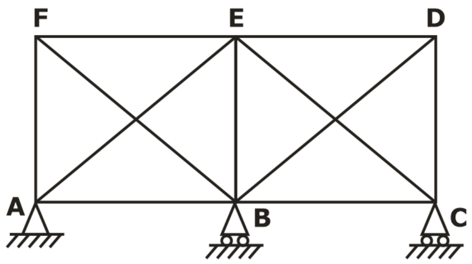
Sol.

Degree of freedom $D_K = 2\times3-5+4$ (Due to internal Hinge) = 5

Ignoring axial deformation, $D_K = 5-2 = 3$

Case 2: Truss

At each joint in a truss number of independent displacements are only two. (horizontal and vertical displacement). Rotation of a member in a truss is not considered because it implies that the member buckled. Rigid body rotation is not counted because it is not unknown.



D_k at A = 0

D_k at B = 1

D_k at C = 1

D_k at D = 2

D_k at E = 2

D_k at F = 2

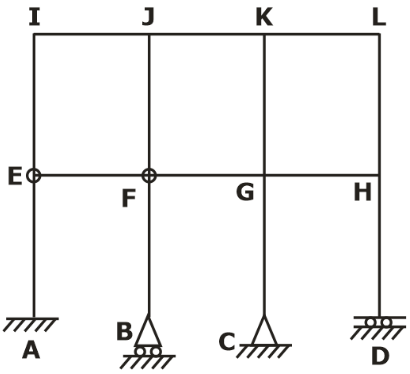
So, degree of freedom

= 0 + 1 + 1 + 2 + 2 + 2 = 8

Case 3: Frames

(i) Count only one rotation for all members meeting at a rigid joint.

(ii) Count rotation of all members meeting at a pin joint.



D_k at A = 0

D_k at B = 2

D_k at C = 1

D_k at D = 1

D_k at E = 5

D_k at F = 6

D_k at G = 3

D_k at H = 3

D_k at I = 3

D_k at J = 3

D_k at K = 3

D_k at L = 3

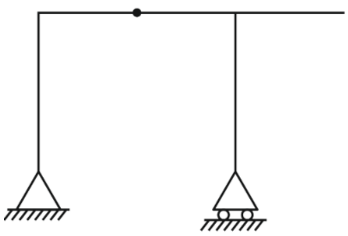
∴ D_k when extensible = 0 + 2 + 1 + 1 + 5 + 6 + 3 + 3 + 3 + 3 + 3 + 3 = 33 degree

D_k when inextensible

= D_k(extensible) – Number of independent displacement prevented.

= 33 – 14 = 29 degrees.

Example:



Sol.

Total degree of freedom D_k= 3×5-3 +4 (Due to internal Hinge) = 16

If members are considered inextensible then,

$$D_k = 16-5=8$$

3. STABILITY OF STRUCTURE

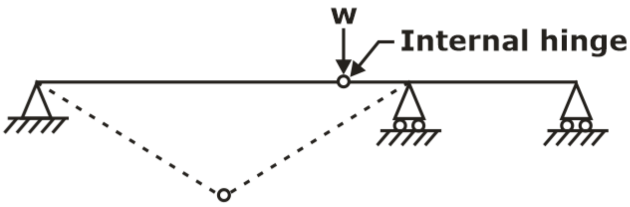
The stability of structure includes external stability and internal stability. The external stability deals with support reaction and internal stability deals within the structure.

3.1. Internal Stability

Internal stability of various cases is explained through the following examples:

Case 1: Beams

→ Internal floating hinge



The above structure is internally unstable.

Case 2: Trusses

In case of trusses if following condition exist then it is classified as unstable truss.

$$\boxed{m < (2j - 3)}$$

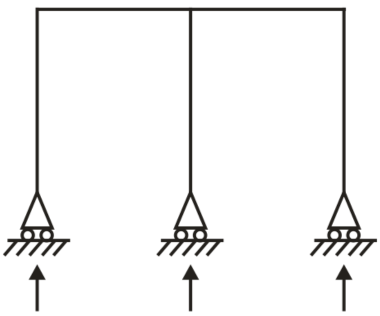
Where,

m = number of members in truss structure.

j = number of joints in truss structure.

Case 3: Frames

If reactions are parallel to each other, then the frame structure will be termed as unstable.



The above shown structure is unstable due to presence of reactions which are parallel.

4. DEFLECTION OF STATICALLY DETERMINATE TRUSSES

Two methods mainly used to calculate deflection in trusses are

(i) Castigliano's Method

(ii) Unit load method

4.1. Castigliano's Method

For getting the deflection in case of truss, there are two theorems. According to these theorem deflection and slope can be determined as follows.

(i) Castigliano's I^st theorem:

$$w = \frac{\partial u}{\partial \delta}$$

Here,

w = load

∂u = change in strain energy

∂δ = variation in deflection.

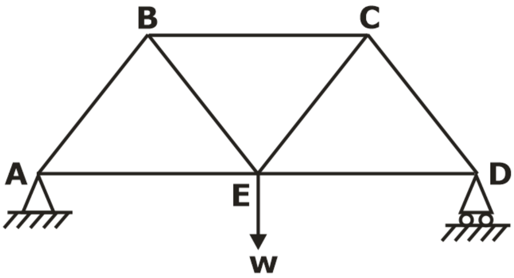
(ii) Castigliano's II^nd theorem

It states, that the first partial derivative of total strain energy with respect to a load at any point in the structure gives deflection at that point in the direction of load.

$$\delta = \frac{\partial u}{\partial P} \qquad \theta = \frac{\partial u}{\partial M}$$

Application of Castigliano's theorem:

(i) To find absolute deflection of a joint in a truss.



U = Strain energy in all members

$$U = \frac{\Sigma P^2 l}{2AE}$$

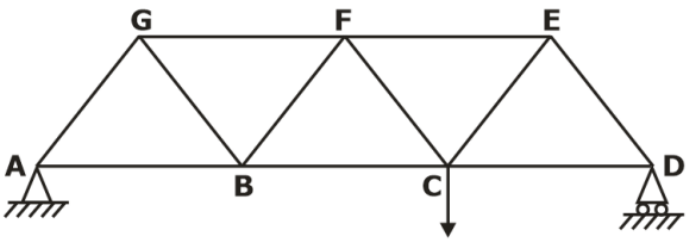
Where, P_1, P_2 … P_n = force in members due to applied load w.

and l_1, l_2 …. L_n = length of each member.

From Castigliano's II theorem

$$\Rightarrow \quad \delta_E = \frac{\partial u}{\partial w} = \Sigma \frac{2P \frac{\partial P}{\partial w} l}{2AE} = \Sigma \frac{Pkl}{AE}$$

Where, $k = k_1, k_2 = \frac{\partial P_1}{\partial w}, \frac{\partial P_2}{\partial w} =$ force in all members due to unit load applied at a point where we have to find deflection (δ).



If we want to find relative displacement of any two joints B and E, apply unit loads at B and E in the direction BE. Find forces in all members due to this load then relative displacement of two joints B and E is

$$\delta_{BE} = \Sigma \frac{Pkl}{AE}$$

Where,

P = P_1, P_2 etc forces in all member due to applied loads unit loads

K = forces in all members due to unit loads applied at B and E.

If we want to find rotation of any member FG, apply unit couple at G and F (these two forces form unit couple i.e. $1/a \times a = 1$). Find forces in all members due to these two loads, then rotation of member is given as

$$\theta_{GF} = \Sigma \frac{Pkl}{AE}$$

Where, P_1, P_2 … P_n = force in all members.

k = forces in all member due to unit couple applied at G and F.

5. IMPORTANT DEFINITIONS

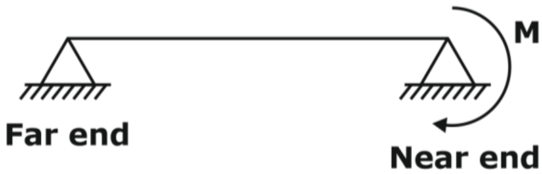
5.1. Stiffness Factor

Stiffness factor can be defined as the moment required to produce unit rotation in the beam. Stiffness factor for various cases is defined as follows.

Case 1: Far end is fixed

$$\text{Stiffness factor} = s = \frac{4EI}{l}$$

Case 2: Far end is hinged



$$Stiffness\ factor = s = \frac{3EI}{l}$$

5.2. Relative stiffness (k)

Relative stiffness is the relative value of the stiffness factor. It is value for various cases can be expressed as follows.

Case 1: Far end is fixed

$$k = \frac{I}{L}$$

Case 2: Far end is hinged

$$k = \frac{3}{4}\frac{I}{L}$$

Case 3: Far end is free

$$K = 0$$

5.3. Distribution Factor

It is the ratio in which the applied moment is distributed to various members meeting at a rigid point. Sum of distribution factor of all members meeting at a rigid joint is one. If far end is free, its D, k and distribution factor is zero.

$$DF = \frac{K}{\Sigma K}$$

Where,

K = Relative stiffness of the member

∑K = Summation of relative stiffness of all members meeting at a joint

5.4. Carry over moment

It is the moment developed at one end due to applied moment at the other end. It is developed to make the slope zero. It is exerted by the fixed support on the beam. It is developed to make slope zero not to keep the structure in equilibrium. Various case for carry over moment are as follows.

Case 1: Far end is fixed

$$COM = \frac{M}{2}$$

Case 2: Far end is hinged

$$COM = 0$$

5.5. Carry Over Factor

Carry over factor can be defined as the ratio of carry over moment and applied moment. Carry over factor for various cases can be given as follows.

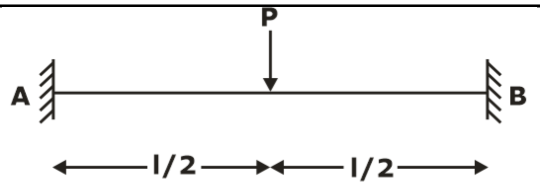
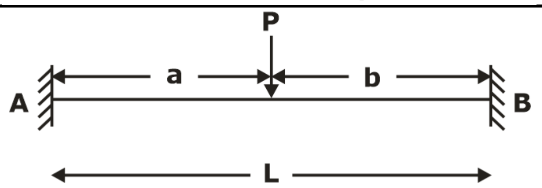
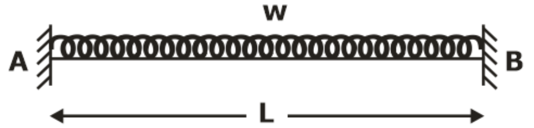
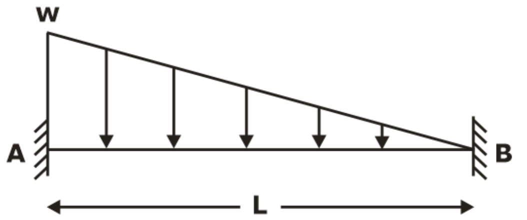
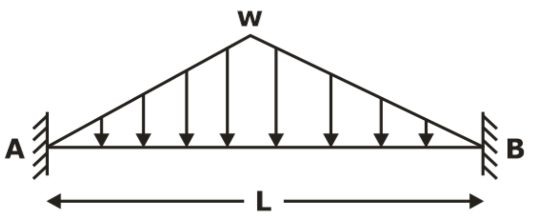
Case 1: Far end is fixed

$$COF = \frac{\frac{M}{2}}{M} = \frac{1}{2}$$

Case 2: Far end is hinged

$$COF = \frac{0}{M} = 0$$

6. FIXED END MOMENTS FOR SOME STANDARD CASES

Beam	Fixed End Moments
	$M_A = M_B = \frac{Pl}{8}$
	$M_A = \frac{Pb^2a}{L}$ $M_B = \frac{Pa^2b}{L}$
	$M_A = \frac{wL^2}{12}$ $M_B = \frac{wL^2}{12}$
	$M_A = \frac{wL^2}{20}$ $M_B = \frac{wL^2}{30}$
	$M_A = \frac{6EI\Delta}{L^2}$ $M_B = \frac{6EI\Delta}{L^2}$
	$M_A = \frac{5wL^2}{96}$ $M_B = \frac{5wL^2}{96}$
	$M_A = \frac{3EI\Delta}{L^2}$, $M_B = 0$

Slope Deflection Equations:

The slope deflection equation can be written as

$$M_{AB} = M_{FAB} + \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B - \frac{6EI\delta}{L^2}$$

$$\Rightarrow M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

And,

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B - \frac{6EI\delta}{L^2}$$

$$\Rightarrow M_{BA} = M_{FBA} + \frac{2EI}{L}\left(\theta_A + 2\theta_B - \frac{3\delta}{L}\right)$$

7. MULLER BRESLAU PRINCIPLE

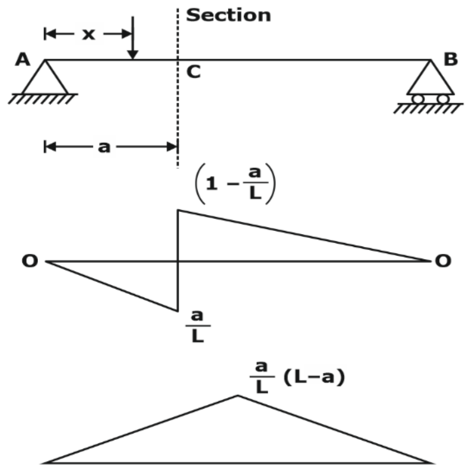
As per this principle, "If an internal stress component or reaction component is considered to act through and tends to deflect a structure than the deflected shape of the structure will be the influence line for the stress or reaction component to some scale."

Note: This Principle gives for quantitative and qualitative deflected shape for determinate structure and qualitative deflected shape for indeterminate structures.

8. MAXIMUM SHEAR FORCE AND BENDING MOMENT FOR A BEAM SUBJECTED TO MOVING LOADS

8.1. Due to Single Point Load

The influence line diagram for a single point load for Shear force and Bending moment is



So, bending moment will be maximum if the load is at the section. For maximum negative shear force the load should be just to the left of section and for maximum positive shear force the load should be just right to the section.

Absolute Maximum shear force and Bending Moment:

Absolute maximum shear force:

For absolute maximum negative shear force, x/L should be maximum. Thus, for absolute maximum negative shear force the value of x would be L i.e. at support B. For absolute maximum positive shear force, $\left(1-\frac{x}{L}\right)$ should be maximum. Thus, for absolute maximum positive shear force the value of x would be zero i.e. at support A.

For absolute maximum bending moment:

$$M_x = \frac{Wx(L-x)}{L}$$

$$\frac{dM_X}{dx} = \frac{W(L-2x)}{L} = 0$$

Thus, absolute maximum bending moment will occur at mid span in case of point load.

8.2. Due to Uniformly Distributed Load Longer than span

Maximum negative shear force occurs when the load covers portion AC only and maximum positive shear force occurs when the load covers the portion CB only. Maximum bending moment at any section will be due to UDL covering the entire span.

Absolute Maximum Value of SF and BM anywhere in the span:

For Absolute maximum Shear Force:

On observing the influence line diagram for Shear force, it is clear that maximum negative shear force will occurs at support B, when the UDL covers the entire span and maximum positive shear force will occurs at support A, when the UDL covers the entire span.

For Absolute maximum bending moment:

Maximum bending moment at any section,

$$M_x = \frac{1}{2} \times w \times \frac{x(L-x)}{L} \times L$$

$$\frac{dM_X}{dx} = \frac{w(L-2x)}{2} = 0$$

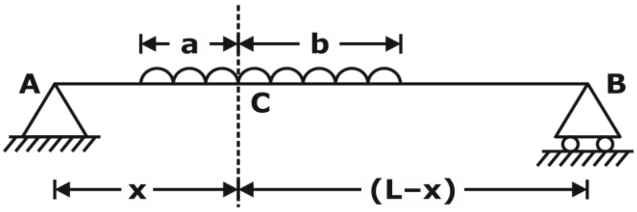
$$\Rightarrow x = \frac{L}{2}$$

Thus, absolute maximum bending moment will occur at mid span in case of UDL larger than the span.

8.3. UDL shorter than the span

For UDL shorter than the span, maximum negative shear force will take place when entire UDL is just left of the section and maximum positive shear force will take place when entire UDL is just right to the section.

For maximum bending moment at C



Load should be placed such that

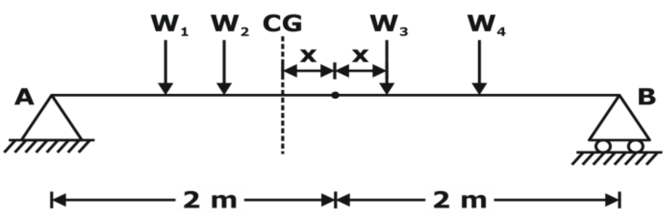
$$\frac{a}{b} = \frac{x}{L-x}$$

8.4. Due to Train of Concentrated Loads

Maximum Bending Moment at a section: Due to train of concentrated loads maximum bending moment will occur at the section if the loads are placed such that the average loading to the left of the section is equal to average loading to the right of the section.

Maximum Bending Moment under a wheel load: Maximum bending moment under a wheel load occurs if the loads are placed such that the load and the resultant of the loading is equidistant from the centre of span.

Maximum bending moment will occur under W_3 if the loads are placed as shown below.



9. TYPES OF ARCHES

There are three types of arches depending upon the number of hinges provided.

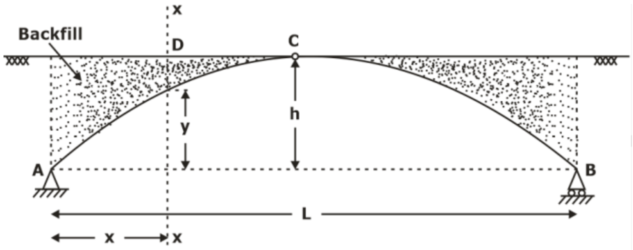
(i) Three hinged arch (Determinate)

(ii) Two hinged arch (Indeterminate to 1 degree)

(iii) Fixed arch (Indeterminate to 3 degree)

9.1. Three hinged Arch

The three hinged arches are statically determinate structure as equations of equilibrium alone are sufficient to find all the unknown quantities.



Circular Arch:

From the properly of a circle the radius r of the circular arch of span L and rise h may be found as

$$\frac{L}{2} \times \frac{L}{2} = h(2R - h)$$

$$\Rightarrow \boxed{R = \frac{L^2}{8h} + \frac{h}{2}}$$

Taking origin at A, the coordinates of any pont d on the arch may be defined as

$$x = \left[\frac{L}{2} - R\sin\theta\right]$$

$$y = R\cos\theta - (R - h)$$

$$\Rightarrow \boxed{y = h - R(1 - \cos\theta)}$$

Parabolic Arch:

Taking spring point as the origin, its equation is given by

$$\boxed{y = \frac{4h_x}{L^2}(L - x)}$$

Bending moment at the section X-X

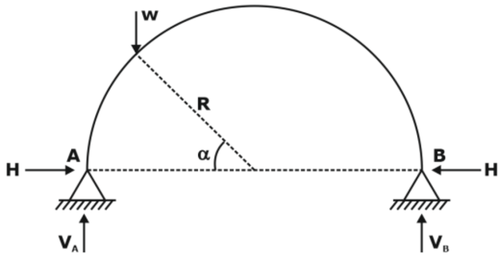
$$BM_{X-X} = +V_A \times x - H_A \times y$$

$$\Rightarrow BM_{X-X} = \text{Beam moment} - \text{H-moment}$$

When compared with a beam of similar span, bending moment at any section in a three hinged arch is less by an amount of '$H \times y$' or moment dur to horizontal force.

9.2. Two hinged arches

A two hinged arch is an indeterminate arch. The horizontal thrust is determined using Castigliano's theorem of least energy.



Two hinged circular arch

Assuming the redundant to be H, As per Castigliano's theorem

$$\frac{\partial U}{\partial H} = 0$$

Which gives the following condition

$$H = \frac{\int_0^l \frac{M_x y dx}{EI_C}}{\int_0^l \frac{y^2 dx}{EI_C}}$$

Where,

M_x = beam moment at any section x – x

I_C = Moment of inertia of the cross section of the arch at the crown.

(i) Horizontal Thrust in case of circular arch subjected to point load

$$H = \frac{W}{\pi}\sin^2\alpha$$

(ii) Horizontal Thrust in case of circular arch subjected to UDL

$$H = \frac{4}{3}\frac{wR}{\pi}$$

(iii) Horizontal Thrust in case of parabolic arch subjected to a point load at centre

$$H = \frac{25}{128}\frac{wL}{H}$$

(iv) Horizontal Thrust in case of parabolic arch subjected to a UDL

$$H = \frac{wl^2}{8h}$$

If there is rib shortening, temperature rise by t°C and yielding of supports then horizontal thrust is given by

$$H = \frac{\int \frac{M_x y dx}{EI_C} + \alpha t l}{\int \frac{y^2 dx}{EI_C} + \frac{l}{AE} + k}$$

Where,

αtl = due to increase in temperature

l/AE due to rib shortening

K = yielding of support/unit horizontal thrust.

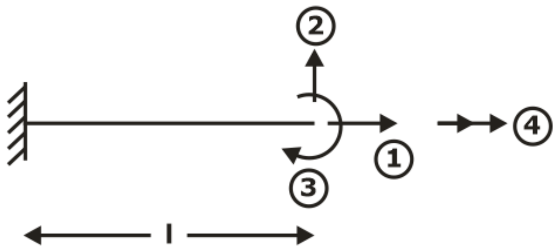
In a two hinged parabolic arch as the temperature increase, horizontal thrust increases. If the effect of rib shortening and yielding of support are considered then horizontal thrust decreases.

10. DISPLACEMENT/STIFFNESS METHOD

In this method displacements at the joints are taken as unknowns and equation are expressed in terms of these unknown displacement. Additional joint equilibrium equations are developed to find the unknown displacement. This method is suitable when the Kinematic indeterminacy is less than the static indeterminacy.

10.1. Stiffness (k)

It is the load required to produce unit displacement. Stiffness for various cases are as follows.



(1) Axial stiffness (k_{11}) = $\frac{AE}{l}$

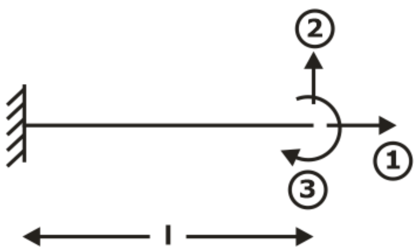
(2) Transverse stiffness (k_{22}) = $\frac{12EI}{l^3}$

(3) Flexural stiffness (k_{33}) = $\frac{4EI}{l}$

(4) Torsional stiffness (k_{44}) = $\frac{GJ}{l}$

10.2. Procedure to Construct Stiffness Matrix

To get first column of stiffness matrix, fix all the coordinates and give unit displacement at the 1^{st} coordinate and find forces developed at all other coordinates similarly to get the second column of stiffness matric apply unit displacement at coordinate 2 and find forces at all coordinates.



The cantilever beam shown in the figure above will be subjected to three displacements (1), (2) and (3).

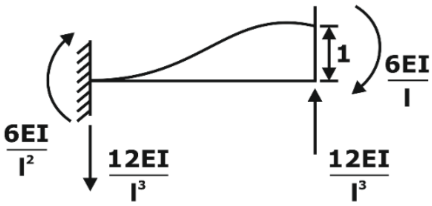
When the unit displacement is given in direction of (1) i.e. horizontal deflection only,

K_{11} = Force at (1) due to unit displacement at (1) = $\frac{AE}{L}$

K_{21} = Force at (2) due to unit displacement at (1) = 0

K_{31} = Force at (3) due to unit displacement at (1) = 0

When the unit displacement is given in direction of (2) i.e. vertical deflection only,

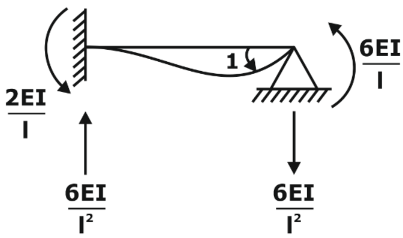


K_{12} = Force at (1) due to unit displacement at (2) = 0

K_{22} = Force at (2) due to unit displacement at (2) = $\frac{2EI}{l^3}$

K_{32} = Force at (3) due to unit displacement at (2) = $-\frac{6EI}{l^2}$

When the unit displacement is given in direction of (3) i.e. rotation only,



K_{13} = Force at (1) due to unit displacement at (3) = 0

K_{23} = Force at (2) due to unit displacement at (3) = $-\frac{6EI}{l^2}$

K_{33} = Force at (3) due to unit displacement at (3) = $\frac{4EI}{l}$

So, the stiffness matrix is

$$K = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{2EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

11. FLEXIBILITY MATRIX METHOD

In this method, forces are taken as unknown and equations are expressed in terms of these forces. Additional equation called compatibility condition are developed to find all the unknown forces. This method is suitable when the static indeterminacy is less than kinematic indeterminacy.

11.1. Flexibility (δ)

Flexibility is defined as the displacement produced due to unit force. It is the inverse of stiffness. Flexibility for various cases are as follows

(a) Axial flexibility $= \frac{1}{\frac{AE}{l}} = \frac{l}{AE}$

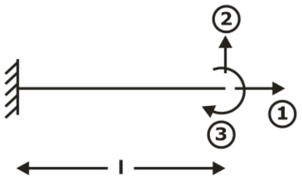
(b) Transverse flexibility $= \frac{1}{\frac{AE}{l^3}} = \frac{l^3}{12AE}$

(c) Flexural flexibility $= \frac{1}{\frac{4AE}{l}} = \frac{l}{4EI}$

(d) Torsional flexibility $= \frac{1}{\frac{GJ}{l}} = \frac{l}{GJ}$

11.2. Procedure to construct Flexibility Matrix

To get the first column of flexibility matrix, apply unit force at coordinate (1) and find displacement at all coordinates in the released structure. Similarly, to get II column of the flexibility matrix apply unit force at coordinate (2) and find displacement at all coordinates in the released structure.



The cantilever beam shown in the figure above is subjected unit forces in three directions.

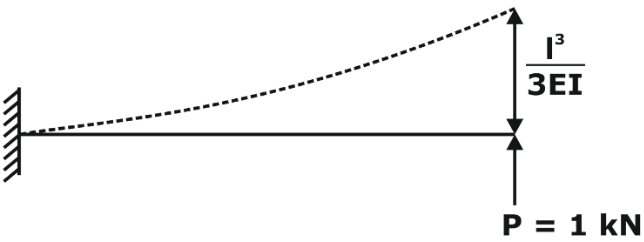
When the unit force is applied in direction of (1)

δ_{11} = displacement at coordinate (1) due to unit load at coordinate (1) $= \frac{l}{AE}$

δ_{21} = displacement at coordinate (2) due to unit load at coordinate (1) = 0

δ_{31} = displacement at coordinate (3) due to unit load at coordinate (1) = 0

When the unit load is applied in the direction of (2)

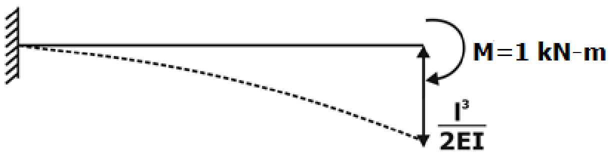


δ_{12} = displacement at coordinate (1) due to unit load at coordinate (2) = 0

δ_{22} = displacement at coordinate (2) due to unit load at coordinate (2) $= \frac{l^3}{3EI}$

δ_{32} = displacement at coordinate (3) due to unit load at coordinate (2) $= -\frac{l^2}{2EI}$

When the unit load is applied in the direction of (3)



δ_{13} = displacement at coordinate (1) due to unit load at coordinate (3) = 0

δ_{23} = displacement at coordinate (2) due to unit load at coordinate (3) = $-\frac{l^2}{2EI}$

δ_{33} = displacement at coordinate (3) due to unit load at coordinate (3) = $\frac{l}{EI}$

So, the flexibility matrix is

$$[\delta] = \begin{bmatrix} \frac{l}{AE} & 0 & 0 \\ 0 & \frac{l^3}{3EI} & -\frac{l^2}{2EI} \\ 0 & -\frac{l^2}{2EI} & \frac{l}{EI} \end{bmatrix}$$

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