

ENGINEERING MATHEMATICS FORMULAS & SHORT NOTES HANDBOOK

Vector Algebra

If i, j, k are orthonormal vectors and $A = A_x i + A_y j + A_z k$ then $|A|^2 = A_x^2 + A_y^2 + A_z^2$. [Orthonormal vectors \equiv orthogonal unit vectors.]

Scalar product

$$A \cdot B = |A| |B| \cos \theta$$

where θ is the angle between the vectors

$$= A_x B_x + A_y B_y + A_z B_z = [A_x A_y A_z] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Scalar multiplication is commutative: $A \cdot B = B \cdot A$.

Equation of a line

A point $r \equiv (x, y, z)$ lies on a line passing through a point a and parallel to vector b if

$$r = a + \lambda b$$

with λ a real number.

Equation of a plane

A point $r \equiv (x, y, z)$ is on a plane if either

(a) $r \cdot \hat{d} = |d|$, where d is the normal from the origin to the plane, or

(b) $\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z} = 1$ where X, Y, Z are the intercepts on the axes.

Vector product

$A \times B = n |A| |B| \sin \theta$, where θ is the angle between the vectors and n is a unit vector normal to the plane containing A and B in the direction for which A, B, n form a right-handed set of axes.

$A \times B$ in determinant form

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$A \times B$ in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative: $A \times B = -B \times A$.

Scalar triple product

$$A \times B \cdot C = A \cdot B \times C = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -A \times C \cdot B, \quad \text{etc.}$$

Vector triple product

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C, \quad (A \times B) \times C = (A \cdot C)B - (B \cdot C)A$$

Non-orthogonal basis

$$A = A_1 e_1 + A_2 e_2 + A_3 e_3$$

$$A_1 = \mathbf{e}' \cdot A \quad \text{where} \quad \mathbf{e}' = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)}$$

Similarly for A_2 and A_3 .

Summation convention

$$\mathbf{a} = a_i \mathbf{e}_i$$

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i$$

$$(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$$

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

implies summation over $i = 1 \dots 3$

where $\varepsilon_{123} = 1$; $\varepsilon_{ijk} = -\varepsilon_{ikj}$

Matrix Algebra

Unit matrices

The unit matrix I of order n is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If A is a square matrix of order n , then $AI = IA = A$. Also $I = I^{-1}$.

I is sometimes written as I_n if the order needs to be stated explicitly.

Products

If A is a $(n \times l)$ matrix and B is a $(l \times m)$ then the product AB is defined by

$$(AB)_{ij} = \sum_{k=1}^l A_{ik}B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If A is a matrix, then transpose matrix A^T is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If A is a square matrix with non-zero determinant, then its inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of A_{ij} is $(-1)^{i+j}$ times the determinant of the matrix A with the j -th row and i -th column deleted.

Determinants

If A is a square matrix then the determinant of A , $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\dots} \epsilon_{ijk\dots} A_{1i}A_{2j}A_{3k}\dots$$

where the number of the suffixes is equal to the order of the matrix.

2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB\dots N)^T = N^T \dots B^T A^T$$

$$(AB\dots N)^{-1} = N^{-1} \dots B^{-1} A^{-1}$$

$$|AB\dots N| = |A| |B| \dots |N|$$

(if individual inverses exist)

(if individual matrices are square)

Orthogonal matrices

An orthogonal matrix Q is a square matrix whose columns q_i form a set of orthonormal vectors. For any orthogonal matrix Q ,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

Solving sets of linear simultaneous equations

If A is square then $Ax = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ if A^{-1} exists, i.e., if $|A| \neq 0$.

If A is square then $Ax = 0$ has a non-trivial solution if and only if $|A| = 0$.

An over-constrained set of equations $Ax = \mathbf{b}$ is one in which A has m rows and n columns, where m (the number of equations) is greater than n (the number of variables). The best solution \mathbf{x} (in the sense that it minimizes the error $|Ax - \mathbf{b}|$) is the solution of the n equations $A^T Ax = A^T \mathbf{b}$. If the columns of A are orthonormal vectors then $\mathbf{x} = A^T \mathbf{b}$.

Hermitian matrices

The Hermitian conjugate of A is $A^\dagger = (A^*)^T$, where A^* is a matrix each of whose components is the complex conjugate of the corresponding components of A . If $A = A^\dagger$ then A is called a Hermitian matrix.

Eigenvalues and eigenvectors

The n eigenvalues λ_i and eigenvectors \mathbf{u}_i of an $n \times n$ matrix A are the solutions of the equation $A\mathbf{u} = \lambda\mathbf{u}$. The eigenvalues are the zeros of the polynomial of degree n , $P_n(\lambda) = |A - \lambda I|$. If A is Hermitian then the eigenvalues λ_i are real and the eigenvectors \mathbf{u}_i are mutually orthogonal. $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A .

$$\text{Tr } A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If S is a symmetric matrix, Λ is the diagonal matrix whose diagonal elements are the eigenvalues of S , and U is the matrix whose columns are the normalized eigenvectors of A , then

$$U^T S U = \Lambda \quad \text{and} \quad S = U \Lambda U^T.$$

If \mathbf{x} is an approximation to an eigenvector of A then $\mathbf{x}^T A \mathbf{x} / (\mathbf{x}^T \mathbf{x})$ (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

$$[A, B] \equiv AB - BA$$

$$[A, B] = -[B, A]$$

$$[A, B]^\dagger = [B^\dagger, A^\dagger]$$

$$[A + B, C] = [A, C] + [B, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Hermitian algebra

$$\mathbf{b}^\dagger = (b_1^*, b_2^*, \dots)$$

	Matrix form	Operator form	Bra-ket form
Hermiticity	$\mathbf{b}^* \cdot A \cdot \mathbf{c} = (A \cdot \mathbf{b})^* \cdot \mathbf{c}$	$\int \psi^* O \phi = \int (O \psi)^* \phi$	$\langle \psi O \phi \rangle$
Eigenvalues, λ real	$A \mathbf{u}_i = \lambda_{(i)} \mathbf{u}_i$	$O \psi_i = \lambda_{(i)} \psi_i$	$O i\rangle = \lambda_i i\rangle$
Orthogonality	$\mathbf{u}_i \cdot \mathbf{u}_j = 0$	$\int \psi_i^* \psi_j = 0$	$\langle i j \rangle = 0 \quad (i \neq j)$
Completeness	$\mathbf{b} = \sum_i \mathbf{u}_i (\mathbf{u}_i \cdot \mathbf{b})$	$\phi = \sum_i \psi_i \left(\int \psi_i^* \phi \right)$	$\phi = \sum_i i\rangle \langle i \phi \rangle$

Rayleigh-Ritz

Lowest eigenvalue	$\lambda_0 \leq \frac{\mathbf{b}^* \cdot A \cdot \mathbf{b}}{\mathbf{b}^* \cdot \mathbf{b}}$	$\lambda_0 \leq \frac{\int \psi^* O \psi}{\int \psi^* \psi}$	$\frac{\langle \psi O \psi \rangle}{\langle \psi \psi \rangle}$
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Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_x \sigma_y = i\sigma_z, \quad \sigma_y \sigma_z = i\sigma_x, \quad \sigma_z \sigma_x = i\sigma_y, \quad \sigma_x \sigma_x = \sigma_y \sigma_y = \sigma_z \sigma_z = I$$

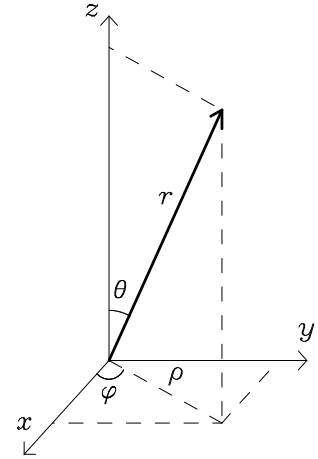
Vector Calculus

Notation

ϕ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \varphi, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. A is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $A = iA_x + jA_y + kA_z$, where A_x, A_y, A_z are independent functions of x, y, z .

$$\text{In Cartesian coordinates } \nabla \text{ ('del')} \equiv i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \equiv \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

$$\text{grad } \phi = \nabla \phi, \quad \text{div } A = \nabla \cdot A, \quad \text{curl } A = \nabla \times A$$



Identities

$$\text{grad}(\phi_1 + \phi_2) \equiv \text{grad } \phi_1 + \text{grad } \phi_2 \quad \text{div}(A_1 + A_2) \equiv \text{div } A_1 + \text{div } A_2$$

$$\text{grad}(\phi_1 \phi_2) \equiv \phi_1 \text{grad } \phi_2 + \phi_2 \text{grad } \phi_1$$

$$\text{curl}(A_1 + A_2) \equiv \text{curl } A_1 + \text{curl } A_2$$

$$\text{div}(\phi A) \equiv \phi \text{div } A + (\text{grad } \phi) \cdot A, \quad \text{curl}(\phi A) \equiv \phi \text{curl } A + (\text{grad } \phi) \times A$$

$$\text{div}(A_1 \times A_2) \equiv A_2 \cdot \text{curl } A_1 - A_1 \cdot \text{curl } A_2$$

$$\text{curl}(A_1 \times A_2) \equiv A_1 \text{div } A_2 - A_2 \text{div } A_1 + (A_2 \cdot \text{grad})A_1 - (A_1 \cdot \text{grad})A_2$$

$$\text{div}(\text{curl } A) \equiv 0, \quad \text{curl}(\text{grad } \phi) \equiv 0$$

$$\text{curl}(\text{curl } A) \equiv \text{grad}(\text{div } A) - \text{div}(\text{grad } A) \equiv \text{grad}(\text{div } A) - \nabla^2 A$$

$$\text{grad}(A_1 \cdot A_2) \equiv A_1 \times (\text{curl } A_2) + (A_1 \cdot \text{grad})A_2 + A_2 \times (\text{curl } A_1) + (A_2 \cdot \text{grad})A_1$$

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad z = z$	$x = r \cos \varphi \sin \theta \quad y = r \sin \varphi \sin \theta$ $z = r \cos \theta$
Vector A	$A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$	$A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z}$	$A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$
Gradient $\nabla \phi$	$\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$	$\frac{\partial \phi}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} \hat{\varphi} + \frac{\partial \phi}{\partial z} \hat{z}$	$\frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{\varphi}$
Divergence $\nabla \cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial A_\theta \sin \theta}{\partial \theta}$ $+ \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$
Curl $\nabla \times A$	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho} \hat{\rho} & \hat{\varphi} & \frac{1}{\rho} \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2 \sin \theta} \hat{r} & \frac{1}{r \sin \theta} \hat{\theta} & \frac{1}{r} \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & r A_\theta & r A_\varphi \sin \theta \end{vmatrix}$
Laplacian $\nabla^2 \phi$	$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right)$ $+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$

Transformation of integrals

L = the distance along some curve 'C' in space and is measured from some fixed point.

S = a surface area

τ = a volume contained by a specified surface

$\hat{\mathbf{t}}$ = the unit tangent to C at the point P

$\hat{\mathbf{n}}$ = the unit outward pointing normal

A = some vector function

dL = the vector element of curve ($= \hat{\mathbf{t}} dL$)

dS = the vector element of surface ($= \hat{\mathbf{n}} dS$)

$$\text{Then } \int_C \mathbf{A} \cdot \hat{\mathbf{t}} dL = \int_C \mathbf{A} \cdot dL$$

and when $A = \nabla \phi$

$$\int_C (\nabla \phi) \cdot dL = \int_C d\phi$$

Gauss's Theorem (Divergence Theorem)

When S defines a closed region having a volume τ

$$\int_\tau (\nabla \cdot \mathbf{A}) d\tau = \int_S (\mathbf{A} \cdot \hat{\mathbf{n}}) dS = \int_S \mathbf{A} \cdot dS$$

also

$$\int_\tau (\nabla \phi) d\tau = \int_S \phi dS$$

$$\int_\tau (\nabla \times \mathbf{A}) d\tau = \int_S (\hat{\mathbf{n}} \times \mathbf{A}) dS$$

Stokes's Theorem

When C is closed and bounds the open surface S ,

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{L}$$

also

$$\int_S (\hat{\mathbf{n}} \times \nabla \phi) \cdot d\mathbf{S} = \int_C \phi \cdot d\mathbf{L}$$

Green's Theorem

$$\begin{aligned} \int_S \psi \nabla \phi \cdot d\mathbf{S} &= \int_{\tau} \nabla \cdot (\psi \nabla \phi) \, d\tau \\ &= \int_{\tau} [\psi \nabla^2 \phi + (\nabla \psi) \cdot (\nabla \phi)] \, d\tau \end{aligned}$$

Green's Second Theorem

$$\int_{\tau} (\psi \nabla^2 \phi - \phi \nabla^2 \psi) \, d\tau = \int_S [\psi (\nabla \phi) - \phi (\nabla \psi)] \cdot d\mathbf{S}$$

5. Complex Variables

Complex numbers

The complex number $z = x + iy = r(\cos \theta + i \sin \theta) = r e^{i(\theta + 2n\pi)}$, where $i^2 = -1$ and n is an arbitrary integer. The real quantity r is the modulus of z and the angle θ is the argument of z . The complex conjugate of z is $z^* = x - iy = r(\cos \theta - i \sin \theta) = r e^{-i\theta}$; $zz^* = |z|^2 = x^2 + y^2$

De Moivre's theorem

$$(\cos \theta + i \sin \theta)^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Power series for complex variables.

e^z	$= 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$	convergent for all finite z
$\sin z$	$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$	convergent for all finite z
$\cos z$	$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$	convergent for all finite z
$\ln(1+z)$	$= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$	principal value of $\ln(1+z)$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots$$

This last series converges both on and within the circle $|z| = 1$ except at the points $z = \pm i$.

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \dots$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

6. Trigonometric Formulae

$$\begin{aligned} \cos^2 A + \sin^2 A &= 1 & \sec^2 A - \tan^2 A &= 1 & \operatorname{cosec}^2 A - \cot^2 A &= 1 \\ \sin 2A &= 2 \sin A \cos A & \cos 2A &= \cos^2 A - \sin^2 A & \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A} \end{aligned}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad \cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \quad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \quad \cos^3 A = \frac{3 \cos A + \cos 3A}{4}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \quad \sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

Relations between sides and angles of any plane triangle

In a plane triangle with angles $A, B,$ and C and sides opposite $a, b,$ and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a = b \cos C + c \cos B$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\text{area} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{1}{2}(a+b+c)$$

Relations between sides and angles of any spherical triangle

In a spherical triangle with angles $A, B,$ and C and sides opposite $a, b,$ and c respectively,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

7. Hyperbolic Functions

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

valid for all x

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

valid for all x

$$\cosh ix = \cos x$$

$$\cos ix = \cosh x$$

$$\sinh ix = i \sin x$$

$$\sin ix = i \sinh x$$

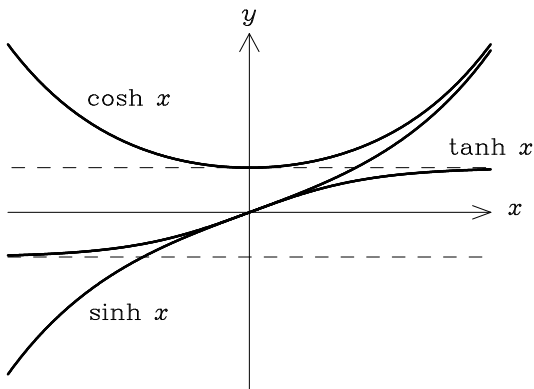
$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$



For large positive x :

$$\cosh x \approx \sinh x \rightarrow \frac{e^x}{2}$$

$$\tanh x \rightarrow 1$$

For large negative x :

$$\cosh x \approx -\sinh x \rightarrow \frac{e^{-x}}{2}$$

$$\tanh x \rightarrow -1$$

Relations of the functions

$$\sinh x = -\sinh(-x)$$

$$\operatorname{sech} x = \operatorname{sech}(-x)$$

$$\cosh x = \cosh(-x)$$

$$\operatorname{cosech} x = -\operatorname{cosech}(-x)$$

$$\tanh x = -\tanh(-x)$$

$$\operatorname{coth} x = -\operatorname{coth}(-x)$$

$$\sinh x = \frac{2 \tanh(x/2)}{1 - \tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}}$$

$$\cosh x = \frac{1 + \tanh^2(x/2)}{1 - \tanh^2(x/2)} = \frac{1}{\sqrt{1 - \tanh^2 x}}$$

$$\tanh x = \sqrt{1 - \operatorname{sech}^2 x}$$

$$\operatorname{sech} x = \sqrt{1 - \tanh^2 x}$$

$$\operatorname{coth} x = \sqrt{\operatorname{cosech}^2 x + 1}$$

$$\operatorname{cosech} x = \sqrt{\operatorname{coth}^2 x - 1}$$

$$\sinh(x/2) = \sqrt{\frac{\cosh x - 1}{2}}$$

$$\cosh(x/2) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh(2x) = 2 \sinh x \cosh x$$

$$\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$\sinh(3x) = 3 \sinh x + 4 \sinh^3 x$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

$$\tanh(3x) = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y) \quad \cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$$

$$\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y) \quad \cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$$

$$\sinh x \pm \cosh x = \frac{1 \pm \tanh(x/2)}{1 \mp \tanh(x/2)} = e^{\pm x}$$

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$

$$\coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

Inverse functions

$$\sinh^{-1} \frac{x}{a} = \ln \left(\frac{x + \sqrt{x^2 + a^2}}{a} \right) \quad \text{for } -\infty < x < \infty$$

$$\cosh^{-1} \frac{x}{a} = \ln \left(\frac{x + \sqrt{x^2 - a^2}}{a} \right) \quad \text{for } x \geq a$$

$$\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln \left(\frac{a + x}{a - x} \right) \quad \text{for } x^2 < a^2$$

$$\coth^{-1} \frac{x}{a} = \frac{1}{2} \ln \left(\frac{x + a}{x - a} \right) \quad \text{for } x^2 > a^2$$

$$\operatorname{sech}^{-1} \frac{x}{a} = \ln \left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1} \right) \quad \text{for } 0 < x \leq a$$

$$\operatorname{cosech}^{-1} \frac{x}{a} = \ln \left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1} \right) \quad \text{for } x \neq 0$$

8. Limits

$$n^c x^n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ if } |x| < 1 \text{ (any fixed } c)$$

$$x^n/n! \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (any fixed } x)$$

$$(1 + x/n)^n \rightarrow e^x \text{ as } n \rightarrow \infty, \quad x \ln x \rightarrow 0 \text{ as } x \rightarrow 0$$

$$\text{If } f(a) = g(a) = 0 \text{ then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \text{ (l'H\^opital's rule)}$$

9. Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \dots + {}^nC_r u^{(n-r)}v^{(r)} + \dots + uv^{(n)}$$

Leibniz Theorem

$$\text{where } {}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

10. Integration

Standard forms

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

for $n \neq -1$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int \ln x dx = x(\ln x - 1) + c$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

$$\int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) + c$$

$$\int x \ln x dx = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + c$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right) + c = \frac{1}{2a} \ln \left(\frac{a+x}{a-x} \right) + c$$

for $x^2 < a^2$

$$\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \coth^{-1} \left(\frac{x}{a} \right) + c = \frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) + c$$

for $x^2 > a^2$

$$\int \frac{x}{(x^2 \pm a^2)^n} dx = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c$$

for $n \neq 1$

$$\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left(x + \sqrt{x^2 \pm a^2} \right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right) \right] + c$$

$$\int_0^{\infty} \frac{1}{(1+x)x^p} dx = \pi \operatorname{cosec} p\pi \quad \text{for } p < 1$$

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) dx = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2) dx = \begin{cases} 1 \times 3 \times 5 \times \dots \times (n-1) \sigma^{n+1} \sqrt{2\pi} & \text{for } n \geq 2 \text{ and even} \\ 0 & \text{for } n \geq 1 \text{ and odd} \end{cases}$$

$$\int \sin x dx = -\cos x + c$$

$$\int \sinh x dx = \cosh x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \cosh x dx = \sinh x + c$$

$$\int \tan x dx = -\ln(\cos x) + c$$

$$\int \tanh x dx = \ln(\cosh x) + c$$

$$\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cot x) + c$$

$$\int \operatorname{cosech} x dx = \ln[\tanh(x/2)] + c$$

$$\int \sec x dx = \ln(\sec x + \tan x) + c$$

$$\int \operatorname{sech} x dx = 2 \tan^{-1}(e^x) + c$$

$$\int \cot x dx = \ln(\sin x) + c$$

$$\int \operatorname{coth} x dx = \ln(\sinh x) + c$$

$$\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \quad \text{if } m^2 \neq n^2$$

$$\int \cos mx \cos nx dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \quad \text{if } m^2 \neq n^2$$

Standard substitutions

If the integrand is a function of: substitute:

$$(a^2 - x^2) \text{ or } \sqrt{a^2 - x^2} \quad x = a \sin \theta \text{ or } x = a \cos \theta$$

$$(x^2 + a^2) \text{ or } \sqrt{x^2 + a^2} \quad x = a \tan \theta \text{ or } x = a \sinh \theta$$

$$(x^2 - a^2) \text{ or } \sqrt{x^2 - a^2} \quad x = a \sec \theta \text{ or } x = a \cosh \theta$$

If the integrand is a rational function of $\sin x$ or $\cos x$ or both, substitute $t = \tan(x/2)$ and use the results:

$$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad dx = \frac{2 dt}{1+t^2}$$

If the integrand is of the form: substitute:

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \quad px+q = u^2$$

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \quad ax+b = \frac{1}{u}$$

Integration by parts

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

Differentiation of an integral

If $f(x, \alpha)$ is a function of x containing a parameter α and the limits of integration a and b are functions of α then

$$\frac{d}{d\alpha} \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) \, dx = f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) \, dx.$$

Special case,

$$\frac{d}{dx} \int_a^x f(y) \, dy = f(x).$$

Dirac δ -'function'

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \tau)] \, d\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t - \tau) f(t) \, dt = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$

Reduction formulae

Factorials

$$n! = n(n-1)(n-2) \dots 1, \quad 0! = 1.$$

Stirling's formula for large n : $\ln(n!) \approx n \ln n - n$.

For any $p > -1$, $\int_0^{\infty} x^p e^{-x} \, dx = p \int_0^{\infty} x^{p-1} e^{-x} \, dx = p!$. $(-1/2)! = \sqrt{\pi}$, $(1/2)! = \sqrt{\pi}/2$, etc.

For any $p, q > -1$, $\int_0^1 x^p (1-x)^q \, dx = \frac{p!q!}{(p+q+1)!}$.

Trigonometrical

If m, n are integers,

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} \theta \cos^n \theta \, d\theta = \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m \theta \cos^{n-2} \theta \, d\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{1}{2}, \quad \int_0^{\pi/2} \sin \theta \, d\theta = 1, \quad \int_0^{\pi/2} \cos \theta \, d\theta = 1, \quad \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

Other

If $I_n = \int_0^{\infty} x^n \exp(-\alpha x^2) \, dx$ then $I_n = \frac{(n-1)}{2\alpha} I_{n-2}$, $I_0 = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$, $I_1 = \frac{1}{2\alpha}$.

11. Differential Equations

Diffusion (conduction) equation

$$\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi$$

Wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Legendre's equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + l(l + 1)y = 0,$$

solutions of which are Legendre polynomials $P_l(x)$, where $P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l$, Rodrigues' formula so $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ etc.

Recursion relation

$$P_l(x) = \frac{1}{l} [(2l - 1)xP_{l-1}(x) - (l - 1)P_{l-2}(x)]$$

Orthogonality

$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{2}{2l + 1} \delta_{ll'}$$

Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0,$$

solutions of which are Bessel functions $J_m(x)$ of order m .

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \quad (\text{integer } m).$$

The same general form holds for non-integer $m > 0$.

Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = [A\rho^n + B\rho^{-n}][C \exp(in\varphi) + D \exp(-in\varphi)]$$

where A, B, C, D are constants and n is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = [Ar^l + Br^{-(l+1)}]P_l^m [C \sin m\varphi + D \cos m\varphi]$$

where l and m are integers with $l \geq |m| \geq 0$; A, B, C, D are constants;

$$P_l^m(\cos \theta) = \sin^{|m|} \theta \left[\frac{d}{d(\cos \theta)} \right]^{|m|} P_l(\cos \theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)[A \cos m\varphi + B \sin m\varphi][C \exp(nz) + D \exp(-nz)]$$

where m and n are integers; A, B, C, D are constants.

Spherical harmonics

The normalized solutions $Y_l^m(\theta, \varphi)$ of the equation

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_l^m + l(l+1)Y_l^m = 0$$

are called spherical harmonics, and have values given by

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^m(\cos \theta) e^{im\varphi} \times \begin{cases} (-1)^m & \text{for } m \geq 0 \\ 1 & \text{for } m < 0 \end{cases}$$

$$\text{i.e., } Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\varphi}, \text{ etc.}$$

Orthogonality

$$\int_{4\pi} Y_l^{*m} Y_{l'}^{m'} d\Omega = \delta_{ll'} \delta_{mm'}$$

12. Calculus of Variations

The condition for $I = \int_a^b F(y, y', x) dx$ to have a stationary value is $\frac{\partial F}{\partial y} = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right)$, where $y' = \frac{dy}{dx}$. This is the Euler-Lagrange equation.

13. Functions of Several Variables

If $\phi = f(x, y, z, \dots)$ then $\frac{\partial \phi}{\partial x}$ implies differentiation with respect to x keeping y, z, \dots constant.

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \dots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x} \delta x + \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial z} \delta z + \dots$$

where x, y, z, \dots are independent variables. $\frac{\partial \phi}{\partial x}$ is also written as $\left(\frac{\partial \phi}{\partial x}\right)_{y, \dots}$ or $\frac{\partial \phi}{\partial x} \Big|_{y, \dots}$ when the variables kept constant need to be stated explicitly.

If ϕ is a well-behaved function then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial \phi}\right)_y}, \quad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + \frac{1}{2!} \left(u^2 \frac{\partial^2 \phi}{\partial x^2} + 2uv \frac{\partial^2 \phi}{\partial x \partial y} + v^2 \frac{\partial^2 \phi}{\partial y^2} \right) + \dots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, y = b$

Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$. Unless $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left. \begin{array}{l} \text{Minimum: } \frac{\partial^2 \phi}{\partial x^2} > 0, \text{ or } \frac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum: } \frac{\partial^2 \phi}{\partial x^2} < 0, \text{ or } \frac{\partial^2 \phi}{\partial y^2} < 0, \end{array} \right\} \text{ and } \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2$$

$$\text{Saddle point: } \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2$$

If $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0$ the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If $\phi = f(x, y, \dots)$ and the variables x, y, \dots are functions of independent variables u, v, \dots then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} + \dots$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial v} + \dots$$

etc.

Changing variables in surface and volume integrals – Jacobians

If an area A in the x, y plane maps into an area A' in the u, v plane then

$$\int_A f(x, y) dx dy = \int_{A'} f(u, v) J du dv \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian J is also written as $\frac{\partial(x, y)}{\partial(u, v)}$. The corresponding formula for volume integrals is

$$\int_V f(x, y, z) dx dy dz = \int_{V'} f(u, v, w) J du dv dw \quad \text{where now} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

14. Fourier Series and Transforms

Fourier series

If $y(x)$ is a function defined in the range $-\pi \leq x \leq \pi$ then

$$y(x) \approx c_0 + \sum_{m=1}^M c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx$$

where the coefficients are

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) dx$$

$$c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos mx dx$$

$$s_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin mx dx$$

$$(m = 1, \dots, M)$$

$$(m = 1, \dots, M')$$

with convergence to $y(x)$ as $M, M' \rightarrow \infty$ for all points where $y(x)$ is continuous.

Fourier series for other ranges

Variable t , range $0 \leq t \leq T$, (i.e., a periodic function of time with period T , frequency $\omega = 2\pi/T$).

$$y(t) \approx c_0 + \sum c_m \cos m\omega t + \sum s_m \sin m\omega t$$

where

$$c_0 = \frac{\omega}{2\pi} \int_0^T y(t) dt, \quad c_m = \frac{\omega}{\pi} \int_0^T y(t) \cos m\omega t dt, \quad s_m = \frac{\omega}{\pi} \int_0^T y(t) \sin m\omega t dt.$$

Variable x , range $0 \leq x \leq L$,

$$y(x) \approx c_0 + \sum c_m \cos \frac{2m\pi x}{L} + \sum s_m \sin \frac{2m\pi x}{L}$$

where

$$c_0 = \frac{1}{L} \int_0^L y(x) dx, \quad c_m = \frac{2}{L} \int_0^L y(x) \cos \frac{2m\pi x}{L} dx, \quad s_m = \frac{2}{L} \int_0^L y(x) \sin \frac{2m\pi x}{L} dx.$$

Fourier series for odd and even functions

If $y(x)$ is an *odd* (anti-symmetric) function [i.e., $y(-x) = -y(x)$] defined in the range $-\pi \leq x \leq \pi$, then only sines are required in the Fourier series and $s_m = \frac{2}{\pi} \int_0^\pi y(x) \sin mx \, dx$. If, in addition, $y(x)$ is symmetric about $x = \pi/2$, then the coefficients s_m are given by $s_m = 0$ (for m even), $s_m = \frac{4}{\pi} \int_0^{\pi/2} y(x) \sin mx \, dx$ (for m odd). If $y(x)$ is an *even* (symmetric) function [i.e., $y(-x) = y(x)$] defined in the range $-\pi \leq x \leq \pi$, then only constant and cosine terms are required in the Fourier series and $c_0 = \frac{1}{\pi} \int_0^\pi y(x) \, dx$, $c_m = \frac{2}{\pi} \int_0^\pi y(x) \cos mx \, dx$. If, in addition, $y(x)$ is anti-symmetric about $x = \frac{\pi}{2}$, then $c_0 = 0$ and the coefficients c_m are given by $c_m = 0$ (for m even), $c_m = \frac{4}{\pi} \int_0^{\pi/2} y(x) \cos mx \, dx$ (for m odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

Complex form of Fourier series

If $y(x)$ is a function defined in the range $-\pi \leq x \leq \pi$ then

$$y(x) \approx \sum_{-M}^M C_m e^{imx}, \quad C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-imx} \, dx$$

with m taking all integer values in the range $\pm M$. This approximation converges to $y(x)$ as $M \rightarrow \infty$ under the same conditions as the real form.

For other ranges the formulae are:

Variable t , range $0 \leq t \leq T$, frequency $\omega = 2\pi/T$,

$$y(t) = \sum_{-\infty}^{\infty} C_m e^{im\omega t}, \quad C_m = \frac{\omega}{2\pi} \int_0^T y(t) e^{-im\omega t} \, dt.$$

Variable x' , range $0 \leq x' \leq L$,

$$y(x') = \sum_{-\infty}^{\infty} C_m e^{i2m\pi x'/L}, \quad C_m = \frac{1}{L} \int_0^L y(x') e^{-i2m\pi x'/L} \, dx'.$$

Discrete Fourier series

If $y(x)$ is a function defined in the range $-\pi \leq x \leq \pi$ which is sampled in the $2N$ equally spaced points $x_n = nx/N$ [$n = -(N-1) \dots N$], then

$$y(x_n) = c_0 + c_1 \cos x_n + c_2 \cos 2x_n + \dots + c_{N-1} \cos(N-1)x_n + c_N \cos Nx_n \\ + s_1 \sin x_n + s_2 \sin 2x_n + \dots + s_{N-1} \sin(N-1)x_n + s_N \sin Nx_n$$

where the coefficients are

$$c_0 = \frac{1}{2N} \sum y(x_n)$$

$$c_m = \frac{1}{N} \sum y(x_n) \cos mx_n \quad (m = 1, \dots, N-1)$$

$$c_N = \frac{1}{2N} \sum y(x_n) \cos Nx_n$$

$$s_m = \frac{1}{N} \sum y(x_n) \sin mx_n \quad (m = 1, \dots, N-1)$$

$$s_N = \frac{1}{2N} \sum y(x_n) \sin Nx_n$$

each summation being over the $2N$ sampling points x_n .

Fourier transforms

If $y(x)$ is a function defined in the range $-\infty \leq x \leq \infty$ then the Fourier transform $\hat{y}(\omega)$ is defined by the equations

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t} d\omega, \quad \hat{y}(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} dt.$$

If ω is replaced by $2\pi f$, where f is the frequency, this relationship becomes

$$y(t) = \int_{-\infty}^{\infty} \hat{y}(f) e^{i2\pi ft} df, \quad \hat{y}(f) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi ft} dt.$$

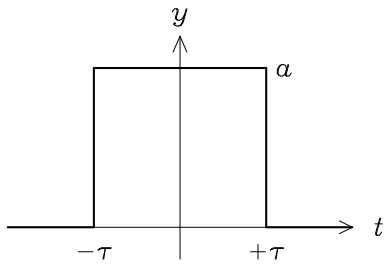
If $y(t)$ is symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi} \int_0^{\infty} \hat{y}(\omega) \cos \omega t d\omega, \quad \hat{y}(\omega) = 2 \int_0^{\infty} y(t) \cos \omega t dt.$$

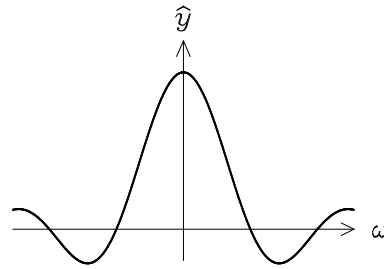
If $y(t)$ is anti-symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi} \int_0^{\infty} \hat{y}(\omega) \sin \omega t d\omega, \quad \hat{y}(\omega) = 2 \int_0^{\infty} y(t) \sin \omega t dt.$$

Specific cases

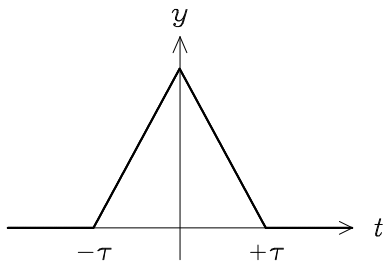


$$y(t) = \begin{cases} a, & |t| \leq \tau \\ 0, & |t| > \tau \end{cases} \quad (\text{'Top Hat'}),$$

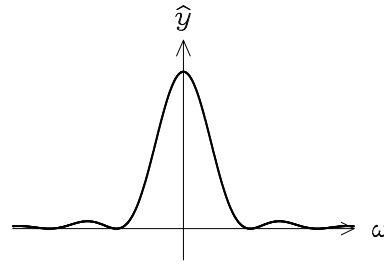


$$\hat{y}(\omega) = 2a \frac{\sin \omega \tau}{\omega} \equiv 2a\tau \operatorname{sinc}(\omega\tau)$$

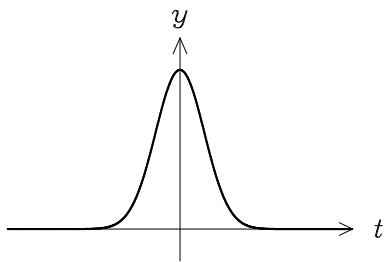
$$\text{where } \operatorname{sinc}(x) = \frac{\sin(x)}{x}$$



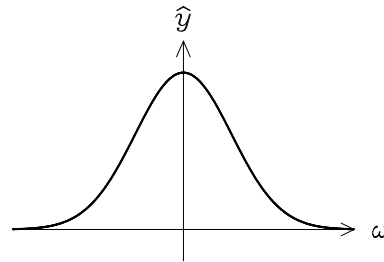
$$y(t) = \begin{cases} a(1 - |t|/\tau), & |t| \leq \tau \\ 0, & |t| > \tau \end{cases} \quad (\text{'Saw-tooth'}),$$



$$\hat{y}(\omega) = \frac{2a}{\omega^2 \tau} (1 - \cos \omega \tau) = a\tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2}\right)$$



$$y(t) = \exp(-t^2/t_0^2) \quad (\text{Gaussian}),$$



$$\hat{y}(\omega) = t_0 \sqrt{\pi} \exp(-\omega^2 t_0^2/4)$$

$$y(t) = f(t) e^{i\omega_0 t} \quad (\text{modulated function}),$$

$$\hat{y}(\omega) = \hat{f}(\omega - \omega_0)$$

$$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau) \quad (\text{sampling function})$$

$$\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)$$

