Mathematics Tips for Air Force Group X & Y: Calculus

Calculus is mainly divided into two parts, viz. differential calculus and integral calculus. While Differential Calculus splits up an area into small parts to calculate the rate of change, Integral calculus joins small parts to calculate the area or volume.

Both the parts have large number of formulae (which you will need to remember). Since formulae makes the most important part of this topic, we will discuss them first (don't get overwhelmed just by looking at the formulae, the more questions you will practice, the easier it will get to remember them).

❖ ***Limit formulae***

1) $\lim_{x\to a} f(x) = f(a)$; where f(x)is a polynomial or rational function in the domain of x.
2) $\frac{0}{0}, \frac{\infty}{\infty}, (\infty - \infty), (\infty \times 0), 1^{\infty}, 0^0, \infty^0$ these all are indeterminate forms.

   When the limit of a rational function has an indeterminate form, then simplify the function by common factors between numerator and denominator, using formulae or by using L'Hospital's rule.
3) L'Hospital's Rule:

   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

   It's good for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms. Derivative is continuous till it doesn't have the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms.
4) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
5) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
6) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$
7) $a^x = 1 + \frac{x \log a}{1!} + \frac{(x \log a)^2}{2!} + \cdots$
8) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
9) $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right]$
10) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
11) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
12) $\lim_{x\to 0} \frac{\sin x}{x} = \cos x = \frac{\tan x}{x} = 1$; (note: here x is in radians and not in degrees)
13) $\lim_{x\to 0} \frac{\sin kx}{x} = \frac{\tan kx}{x} = k$
14) $\lim_{x\to \infty} \frac{\sin x}{x} = \frac{\tan x}{x} = 0$

15) $\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx} = \frac{a^2}{b^2}$

16) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{\tan ax}{\tan bx} = \frac{a}{b}$

17) $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

18) $\sum n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

19) $\sum n^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

20) $\sum n^3 = 1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

❖ ***Differentiation formulae:***

1) $\frac{d}{dx}(k) = 0$; where k is a constant

2) $\frac{d}{dx}(k.u) = k.\frac{du}{dx}$; where k is a constant and u is a function of x.

3) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$; where u and v are functions of x.

4) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

5) $\frac{d}{dx}(u.v.w) = vw.\frac{du}{dx} + uw.\frac{dv}{dx} + uv.\frac{dw}{dx}$; where u, v and w are functions of x.

6) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$

7) If y is a differentiable function of u and u is a differentiable function of x, then

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ **(Chain Rule)**

8) If x and y both are functions of the same variable (t), i.e. y=*f (t)* and x=*g (t),* then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

9) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$; it is called second derivative.

If *y = f(x)*, then $\frac{d^2y}{dx^2}$ can also be written as *f''(x) or* $f^2(x)$.

10) $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$; it is called third derivative.

11) $\frac{d}{dx}x^n = nx^{n-1}$

12) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$

13) $\frac{d}{dx}(\sin x) = \cos x$

14) $\frac{d}{dx}(\cos x) = -\sin x$

15) $\frac{d}{dx}(\tan x) = sec^2 x$

16) $\frac{d}{dx}(\cot x) = -cosec^2\, x$

17) $\frac{d}{dx}(\sec x) = \sec x \tan x$

18) $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$

19) $\frac{d}{dx}(e^x) = e^x$

20) $\frac{d}{dx}(a^x) = a^x \ln a$

21) $\frac{d}{dx}(\ln \mathrm{x}) = \frac{1}{x}$

22) $\frac{d}{dx}(sin^{-1}\, \mathrm{x}) = \frac{1}{\sqrt{1-x^2}}$

23) $\frac{d}{dx}(cos^{-1}\, \mathrm{x}) = \frac{-1}{\sqrt{1-x^2}}$

24) $\frac{d}{dx}(tan^{-1}\, \mathrm{x}) = \frac{1}{1+x^2}$

25) $\frac{d}{dx}(cot^{-1}\, \mathrm{x}) = \frac{-1}{1+x^2}$

26) $\frac{d}{dx}(sec^{-1}\, \mathrm{x}) = \frac{1}{x\sqrt{x^2-1}}$

27) $\frac{d}{dx}(cosec^{-1}\, \mathrm{x}) = \frac{-1}{\sqrt{1-x^2}}$

<u>Integration formulae</u>

1) $\int a\, dx = ax + c$
2) $\int k.f(x)\, dx = k.\int f(x)dx$; where k is a constant.
3) $\int (u \pm v \pm w \pm \cdots)dx = \int u\, dx \pm \int v\, dx \pm \int w\, dx \pm \cdots$; where u, v, w ... are functions of x.
4) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$; here $x \neq -1$
5) $\int \frac{1}{x} dx = \ln|x| + c$
6) $\int e^x dx = e^x + c$
7) $\int a^x dx = \frac{a^x}{\ln a} + c$
8) $\int \ln x\, dx = x \ln x - x + c$
9) $\int \sin x\, dx = -\cos x + c$
10) $\int \cos x\, dx = \sin x + c$
11) $\int tanx\, dx = -\ln|\cos x| + c = \ln|\sec x| + c$
12) $\int \cot x\ \ dx = \ln|\sin x| + c = -\ln|cosec\ x| + c$
13) $\int \sec x\ \ dx = \ln|\sec x + \tan x| + c$

14) $\int cosec\ x\ \ dx = \ln|cosec\ x - \cot x| + c$

15) $\int sec^2 x\ dx = \tan x + c$

16) $\int \sec x \tan x\ dx = \sec x + c$

17) $\int cosec^2 x\ dx = -\cot x + c$

18) $\int cosec\ x\ .\cot x\ dx = -cosec\ x + c$

19) $\int tan^2 x\ dx = \tan x - x + c$

20) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c \Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

21) $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\frac{x}{a} + c \Rightarrow \int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$

22) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \Rightarrow \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

23) $\int \frac{-1}{a^2+x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + c \Rightarrow \int \frac{-1}{1+x^2} dx = \cot^{-1} x + c$

24) $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \Rightarrow \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

25) $\int \frac{-1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}cosec^{-1}\left(\frac{x}{a}\right) + c \Rightarrow \int \frac{-1}{x\sqrt{x^2-1}} dx = cosec^{-1} x + c$

❖ Tips and Tricks

- Level of questions is not that tough so try not to waste your energy on very difficult questions. Instead invest your time in understanding the basic concepts and familiarising yourself with the different variety of questions.
- Questions in calculus are tricky and need a lot of practice. They generally cannot be solved by putting values or using shortcuts. So solve as much questions as you can.
- Remember the formulae, it is impossible to solve questions in exam without them. Differentiation and integration are bidirectional and their formulae can be remembered by remembering any one of them. We have covered all the important calculus formulae here in this article. Revise them from here time to time (and before exam).
- When the direct integration of a function is difficult, you can use these three methods of integration:

I. **Transformation method**:

Eg: $\int \sqrt{1-\sin 2\theta}\ \ d\theta = \int \sqrt{sin^2\theta + cos^2\theta - 2sin\theta.cos\theta}\ d\theta$

$= \sqrt{(cos\theta - sin\theta)^2}\ \ d\theta = \int (cos\theta - sin\theta)\ d\theta = sin\theta + cos\theta + c$

II. **Substitution method:**

Eg: $\int sec^2(2x+3)dx$

Let (2x+3) =z, then dz=2dx or dx=$\frac{dz}{2}$

$= \int sec^2 z.\frac{dz}{2} = \frac{1}{2}\int sec^2 z\ dz = \frac{1}{2}\tan z + c$

$= \frac{1}{2}\tan(2x+3) + c$

III. **Integration by parts:**

This method is used when integrand is the product of two functions.

Eg: $\int u.v.dx = u.\int v\,dx - \int\left(\frac{du}{dx}\int v dx\right)dx$

To decide which of the function should be taken first (as u), you should give priority according to ILATE formula, where

I→ Inverse circular function

L→ Logrithmic function

A→ Algebraic function

T→ Trignometric function

E→ Exponential function

Eg: $\int \tan^{-1}x\ dx$

$$\int \tan^{-1}x\,.x^0\,dx = \tan^{-1}x\int x^0dx - \int\left(\frac{d}{dx}\tan^{-1}x\int x^0dx\right)dx$$

$$= x\tan^{-1}x - \int\frac{1}{1+x^2}.x\,dx = x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + c$$

❖ **Exemplar Problems:**

1. $\lim\limits_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x} = ?$

(A) $\frac{1}{2}\cos 2$ **(B)1** **(C)** $2\cos 2$ **(D)0**

Solution:

Using formula sin(a+b) = sin a cos b + cos a sin b

⇨ $\lim\limits_{x\to 0}\frac{\sin 2\cos x+\cos 2\sin x-\sin 2\cos x+\cos 2\sin x}{x}$

⇨ $\lim\limits_{x\to 0}\frac{2\cos 2\sin x}{x}$

⇨ $2\cos 2\lim\limits_{x\to 0}\frac{sinx}{x}$

Here we know that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

⇨ So, the answer will be (c) 2cos2

2. $\int\frac{\cos 2x}{cos^2xsin^2x}dx = ?$

(A) $\cot x + \tan x + c$ **(B)** $\cot x - \tan x + c$

(C) $-\cot x - \tan x + c$ **(D)** $\tan x - \cot x + c$

Solution:

⇨ $\int\frac{cos^2x-sin^2x}{cos^2xsin^2x}dx$ [since cos 2x = $\cos^2$x – $\sin^2$x]

⇨ $\int \frac{cos^2x}{cos^2xsin^2x} - \frac{sin^2x}{cos^2xsin^2x}\ dx$

⇨ $\int \frac{1}{sin^2x} - \frac{1}{cos^2x}\ dx$

⇨ $\int (cosec^2x - sec^2x)\ dx$

⇨ $-\cot x - \tan x + c$

⇨ So the correct answer is (C)

3. Find the solution of the differential equation $\frac{dy}{dx} = e^{x+y} + x^2e^y$

(A) $e^x - e^y + \frac{y^3}{3} = c$ **(B)** $e^x + e^y + \frac{x^3}{3} = c$

(C) $e^x + e^{-y} + \frac{x^3}{3} = c$ **(D)** $e^x + e^{-y} + \frac{y^3}{3} = c$

Solution:

⇨ $\frac{dy}{dx} = e^y(e^x + x^2)$

⇨ $\frac{dy}{e^y} = (e^x + x^2)dx$

⇨ Integrating both sides

$$\int e^{-y}dy = \int (e^x + x^2)dx$$

⇨ $-e^{-y} + c = e^x + \frac{x^3}{3} + c$

⇨ $e^x + e^{-y} + \frac{x^3}{3} = c$

⇨ So the correct answer is (C).

4. $\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \boldsymbol{sin^2x\ dx} = ?$

(A) 1 **(B)** $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ **(C)** $\frac{\pi}{2} - \frac{1}{4}$ **(D) 0**

Solution:

Using cos 2x = 1 – 2sin^{2}x

⇨ $\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \frac{1-\cos 2x}{2}\ dx$

⇨ $\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} (1 - \cos 2x)\ dx$

⇨ $\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]$

⇨ Putting the limit

⇨ $\frac{1}{2}\left[\frac{\pi}{3}-\frac{\sin\frac{2\pi}{3}}{2}+\frac{\pi}{3}+\frac{\sin\frac{-2\pi}{3}}{2}\right]$

⇨ $\frac{1}{2}\left[\frac{2\pi}{3}-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}\right]$

⇨ $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

⇨ Hence the correct answer is (B).

5. If $f(x)=\log\left(x+\sqrt{x^2+1}\right)$ then find $f'(x)$=?

(A) $\frac{-1}{\sqrt{x^2-1}}$ **(B)** $\frac{1}{\sqrt{x^2-1}}$ **(C)** $\frac{1}{\sqrt{x^2+1}}$ **(D)** $\frac{-1}{\sqrt{x^2+1}}$

Solution:

Using Chain Rule

$$f'(x)=\left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1+\frac{1\,.\,2x}{2\sqrt{x^2+1}}\right)$$

$$f'(x)=\left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$f'(x)=\frac{1}{\sqrt{x^2+1}}$$

Hence the correct answer is (C).