Complex Numbers

Argument of Complex Number

It is the angle made by the line segment joining the complex number and origin in the Argand plane with the positive direction of real axis in anticlockwise direction.

If argument lies in the interval (–π, π] it is known as principle argument. It is denoted by Arg(z).

Argument of a complex number = Principle argument + 2nπ : n єz.

arg(z) = Arg(z) + 2nπ

Arg(x+ iy) =	$\tan^{-1}\frac{y}{x}$	; If	$x > 0$		
	$\tan^{-1}\frac{y}{x}+\pi$	; If	$x < 0$	and	$y \geq 0$
	$\tan^{-1}\frac{y}{x}-\pi$	; If	$x < 0$	and	$y < 0$
	$+\frac{\pi}{2}$	; If	$x = 0$	and	$y > 0$
	$-\frac{\pi}{2}$	; If	$x = 0$	and	$y < 0$
	Undefined	; If	$x = y = 0$		

Polar Form of A Complex Number: Any complex number z = x + iy can be written as

$z = r(\cos\theta + i\sin\theta)$; where $r = |z|$;

θ = argument of z = arg(z)

Euler's Form:

Any complex number $z = r(\cos\theta + i\sin\theta)$ can be written as $z = re^{i\theta}$;

where $\cos\theta + i\sin\theta = e^{i\theta}$

Properties of Argument:

(1) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(2) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

(3) $\arg(z)^n = n \arg z$

(4) $\arg(1/z) = -\arg z$

However $\text{Arg}(z) \in (-\pi, \pi]$ so its properties are

(1) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2\pi N_+$

(2) $\text{Arg}(z_1/z_2) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2\pi N_-$

$$N_\pm = \begin{cases} -1 & \text{if} \quad \text{Arg}(z_1) \pm \text{Arg}(z_2) > \pi \\ 0 & \text{if} \quad -\pi < \text{Arg}(z_1) \pm \text{Arg}(z_2) \le \pi \\ 1 & \text{if} \quad \text{Arg}(z_1) \pm \text{Arg}(z_2) \le -\pi \end{cases}$$

(3) $\text{Arg}(z^n) = n\,\text{Arg}(z) + 2\pi k$

De-Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z^n = r^n (\cos\theta + i\sin\theta)^n = r^n (e^{i\theta})^n$

$= r^n e^{in\theta} = r^n (\cos n\theta + i\sin n\theta)$

$\therefore \quad (\cos\theta \pm i\sin\theta)^n = \text{con } n\theta \pm i\sin n\theta$

(only valid for polar/Euler form)

Q1. For a non-zero complex number z, let Arg(z) denote the principle argument with $-\pi < \text{Arg}(z) \le \pi$. Then,

A. $\arg(-1 - i) = \pi/4$

B. the function $f: R \to (-\pi, \pi]$ defined by $f(t) = \text{Arg}(-1 + it)$ for all $t \in R$ is continuous at all points of R.

C. For only two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integral multiple of π.

D. $\arg\left(\left(1 + \cos\frac{6\pi}{5}\right) + i\sin\frac{6\pi}{5}\right) = \frac{-2\pi}{5}$

Q2. Let a, b ϵR and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{ z \in C : z = \frac{1}{a+ibt}, t \in R;\ t \neq 0 \right\}$

If z = x + iy and Z ϵ S, then (x, y) lies on

A. the circle with radius 1/2a and centre $\left(\frac{1}{2a}, 0\right)$ for a > 0; b ≠ 0

B. the circle with radius $\frac{-1}{2a}$ and centre $\left(\frac{-1}{2a}, 0\right)$ for a < 0; b ≠ 0

C. the x-axis for a ≠ 0; b = 0

D. the y-axis for a = 0; b ≠ 0

Q3. If |z| = 1, find the locus of the curve.

(i) $\frac{1}{1-z}$

(ii) $\frac{z}{1-z^2}$

Q4. Let z = cosθ + isinθ. Then value of $\sum_{m=1}^{15} \text{Im}\left(z^{2m-1}\right)$ at θ = 2° is

A. $\frac{1}{\sin 2°}$

B. $\frac{1}{3\sin 2°}$

C. $\frac{1}{2\sin 2°}$

D. $\frac{1}{4\sin 2°}$

Q5. Find the point of intersection of the curves represented by

arg(z – 3i) $= \frac{3\pi}{4}$ and arg(2z + 1 – 2i) $= \frac{\pi}{4}$.

Q6. If z_1 and z_2 are the roots of the equation $z^2 - 2z + 4 = 0$ and if $z_1^{11} + z_2^{11} = 2^k$. Find k.

Q7. Let $W = \frac{\sqrt{3} + i}{2}$ and P = {w^n : n = 1, 2, 3, ….} Further $H_1 = \left\{z \in C : Re(z) > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : Re(z) > \frac{-1}{2}\right\}$ where C is set of all complex Numbers.

If $z_1 \in P \cap H_1$ and $z_2 \in P \cap H_2$. Find $\angle z_1 O z_2$, where O is the origin.

Q8. If $|z| \le 1$; $|w| \le 1$

Prove that $|z - w|^2 \le (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$