



# Engineering Mathematics

## Important Formulas

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# ENGINEERING MATHEMATICS FORMULAS & SHORT NOTES HANDBOOK

## Vector Algebra

If  $i, j, k$  are orthonormal vectors and  $A = A_x i + A_y j + A_z k$  then  $|A|^2 = A_x^2 + A_y^2 + A_z^2$ . [Orthonormal vectors  $\equiv$  orthogonal unit vectors.]

### Scalar product

$$A \cdot B = |A| |B| \cos \theta$$

where  $\theta$  is the angle between the vectors

$$= A_x B_x + A_y B_y + A_z B_z = [A_x A_y A_z] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Scalar multiplication is commutative:  $A \cdot B = B \cdot A$ .

### Equation of a line

A point  $r \equiv (x, y, z)$  lies on a line passing through a point  $a$  and parallel to vector  $b$  if

$$r = a + \lambda b$$

with  $\lambda$  a real number.

## Equation of a plane

A point  $r \equiv (x, y, z)$  is on a plane if either

(a)  $r \cdot \hat{d} = |d|$ , where  $d$  is the normal from the origin to the plane, or

(b)  $\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z} = 1$  where  $X, Y, Z$  are the intercepts on the axes.

## Vector product

$A \times B = n |A| |B| \sin \theta$ , where  $\theta$  is the angle between the vectors and  $n$  is a unit vector normal to the plane containing  $A$  and  $B$  in the direction for which  $A, B, n$  form a right-handed set of axes.

$A \times B$  in determinant form

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$A \times B$  in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative:  $A \times B = -B \times A$ .

## Scalar triple product

$$A \times B \cdot C = A \cdot B \times C = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -A \times C \cdot B, \text{ etc.}$$

## Vector triple product

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C, \quad (A \times B) \times C = (A \cdot C)B - (B \cdot C)A$$

## Non-orthogonal basis

$$A = A_1 e_1 + A_2 e_2 + A_3 e_3$$

$$A_1 = e'_1 \cdot A \quad \text{where} \quad e'_1 = \frac{e_2 \times e_3}{e_1 \cdot (e_2 \times e_3)}$$

Similarly for  $A_2$  and  $A_3$ .

## Summation convention

$$\mathbf{a} = a_i \mathbf{e}_i$$

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i$$

$$(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$$

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

implies summation over  $i = 1 \dots 3$

where  $\varepsilon_{123} = 1$ ;  $\varepsilon_{ijk} = -\varepsilon_{ikj}$

## Matrix Algebra

### Unit matrices

The unit matrix  $I$  of order  $n$  is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e.,  $(I)_{ij} = \delta_{ij}$ . If  $A$  is a square matrix of order  $n$ , then  $AI = IA = A$ . Also  $I = I^{-1}$ .

$I$  is sometimes written as  $I_n$  if the order needs to be stated explicitly.

### Products

If  $A$  is a  $(n \times l)$  matrix and  $B$  is a  $(l \times m)$  then the product  $AB$  is defined by

$$(AB)_{ij} = \sum_{k=1}^l A_{ik}B_{kj}$$

In general  $AB \neq BA$ .

### Transpose matrices

If  $A$  is a matrix, then transpose matrix  $A^T$  is such that  $(A^T)_{ij} = (A)_{ji}$ .

### Inverse matrices

If  $A$  is a square matrix with non-zero determinant, then its inverse  $A^{-1}$  is such that  $AA^{-1} = A^{-1}A = I$ .

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of  $A_{ij}$  is  $(-1)^{i+j}$  times the determinant of the matrix  $A$  with the  $j$ -th row and  $i$ -th column deleted.

### Determinants

If  $A$  is a square matrix then the determinant of  $A$ ,  $|A|$  ( $\equiv \det A$ ) is defined by

$$|A| = \sum_{i,j,k,\dots} \epsilon_{ijk\dots} A_{1i}A_{2j}A_{3k}\dots$$

where the number of the suffixes is equal to the order of the matrix.

### 2×2 matrices

If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then,

$$|A| = ad - bc \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

### Product rules

$$(AB\dots N)^T = N^T \dots B^T A^T$$

$$(AB\dots N)^{-1} = N^{-1} \dots B^{-1} A^{-1}$$

$$|AB\dots N| = |A| |B| \dots |N|$$

(if individual inverses exist)

(if individual matrices are square)

### Orthogonal matrices

An orthogonal matrix  $Q$  is a square matrix whose columns  $q_i$  form a set of orthonormal vectors. For any orthogonal matrix  $Q$ ,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

## Solving sets of linear simultaneous equations

If  $A$  is square then  $Ax = b$  has a unique solution  $x = A^{-1}b$  if  $A^{-1}$  exists, i.e., if  $|A| \neq 0$ .

If  $A$  is square then  $Ax = 0$  has a non-trivial solution if and only if  $|A| = 0$ .

An over-constrained set of equations  $Ax = b$  is one in which  $A$  has  $m$  rows and  $n$  columns, where  $m$  (the number of equations) is greater than  $n$  (the number of variables). The best solution  $x$  (in the sense that it minimizes the error  $|Ax - b|$ ) is the solution of the  $n$  equations  $A^T Ax = A^T b$ . If the columns of  $A$  are orthonormal vectors then  $x = A^T b$ .

## Hermitian matrices

The Hermitian conjugate of  $A$  is  $A^\dagger = (A^*)^T$ , where  $A^*$  is a matrix each of whose components is the complex conjugate of the corresponding components of  $A$ . If  $A = A^\dagger$  then  $A$  is called a Hermitian matrix.

## Eigenvalues and eigenvectors

The  $n$  eigenvalues  $\lambda_i$  and eigenvectors  $u_i$  of an  $n \times n$  matrix  $A$  are the solutions of the equation  $Au = \lambda u$ . The eigenvalues are the zeros of the polynomial of degree  $n$ ,  $P_n(\lambda) = |A - \lambda I|$ . If  $A$  is Hermitian then the eigenvalues  $\lambda_i$  are real and the eigenvectors  $u_i$  are mutually orthogonal.  $|A - \lambda I| = 0$  is called the characteristic equation of the matrix  $A$ .

$$\text{Tr } A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If  $S$  is a symmetric matrix,  $\Lambda$  is the diagonal matrix whose diagonal elements are the eigenvalues of  $S$ , and  $U$  is the matrix whose columns are the normalized eigenvectors of  $A$ , then

$$U^T S U = \Lambda \quad \text{and} \quad S = U \Lambda U^T.$$

If  $x$  is an approximation to an eigenvector of  $A$  then  $x^T A x / (x^T x)$  (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

## Commutators

$$[A, B] \equiv AB - BA$$

$$[A, B] = -[B, A]$$

$$[A, B]^\dagger = [B^\dagger, A^\dagger]$$

$$[A + B, C] = [A, C] + [B, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

## Hermitian algebra

$$b^\dagger = (b_1^*, b_2^*, \dots)$$

	Matrix form	Operator form	Bra-ket form
Hermiticity	$b^* \cdot A \cdot c = (A \cdot b)^* \cdot c$	$\int \psi^* O \phi = \int (O \psi)^* \phi$	$\langle \psi   O   \phi \rangle$
Eigenvalues, $\lambda$ real	$A u_i = \lambda_{(i)} u_i$	$O \psi_i = \lambda_{(i)} \psi_i$	$O  i\rangle = \lambda_i  i\rangle$
Orthogonality	$u_i \cdot u_j = 0$	$\int \psi_i^* \psi_j = 0$	$\langle i   j \rangle = 0 \quad (i \neq j)$
Completeness	$b = \sum_i u_i (u_i \cdot b)$	$\phi = \sum_i \psi_i \left( \int \psi_i^* \phi \right)$	$\phi = \sum_i  i\rangle \langle i   \phi \rangle$

### Rayleigh-Ritz

Lowest eigenvalue	$\lambda_0 \leq \frac{b^* \cdot A \cdot b}{b^* \cdot b}$	$\lambda_0 \leq \frac{\int \psi^* O \psi}{\int \psi^* \psi}$	$\frac{\langle \psi   O   \psi \rangle}{\langle \psi   \psi \rangle}$
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## Pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_x \sigma_y = i\sigma_z, \quad \sigma_y \sigma_z = i\sigma_x, \quad \sigma_z \sigma_x = i\sigma_y, \quad \sigma_x \sigma_x = \sigma_y \sigma_y = \sigma_z \sigma_z = I$$

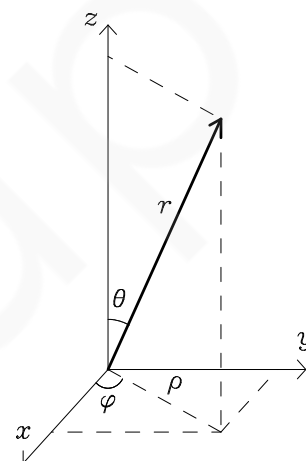
## Vector Calculus

### Notation

$\phi$  is a scalar function of a set of position coordinates. In Cartesian coordinates  $\phi = \phi(x, y, z)$ ; in cylindrical polar coordinates  $\phi = \phi(\rho, \phi, z)$ ; in spherical polar coordinates  $\phi = \phi(r, \theta, \phi)$ ; in cases with radial symmetry  $\phi = \phi(r)$ .  $A$  is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates  $A = iA_x + jA_y + kA_z$ , where  $A_x, A_y, A_z$  are independent functions of  $x, y, z$ .

In Cartesian coordinates  $\nabla$  ('del')  $\equiv i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \equiv \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$

$$\text{grad } \phi = \nabla \phi, \quad \text{div } A = \nabla \cdot A, \quad \text{curl } A = \nabla \times A$$



### Identities

$$\text{grad}(\phi_1 + \phi_2) \equiv \text{grad } \phi_1 + \text{grad } \phi_2 \quad \text{div}(A_1 + A_2) \equiv \text{div } A_1 + \text{div } A_2$$

$$\text{grad}(\phi_1 \phi_2) \equiv \phi_1 \text{grad } \phi_2 + \phi_2 \text{grad } \phi_1$$

$$\text{curl}(A_1 + A_2) \equiv \text{curl } A_1 + \text{curl } A_2$$

$$\text{div}(\phi A) \equiv \phi \text{div } A + (\text{grad } \phi) \cdot A, \quad \text{curl}(\phi A) \equiv \phi \text{curl } A + (\text{grad } \phi) \times A$$

$$\text{div}(A_1 \times A_2) \equiv A_2 \cdot \text{curl } A_1 - A_1 \cdot \text{curl } A_2$$

$$\text{curl}(A_1 \times A_2) \equiv A_1 \text{div } A_2 - A_2 \text{div } A_1 + (A_2 \cdot \text{grad})A_1 - (A_1 \cdot \text{grad})A_2$$

$$\text{div}(\text{curl } A) \equiv 0, \quad \text{curl}(\text{grad } \phi) \equiv 0$$

$$\text{curl}(\text{curl } A) \equiv \text{grad}(\text{div } A) - \text{div}(\text{grad } A) \equiv \text{grad}(\text{div } A) - \nabla^2 A$$

$$\text{grad}(A_1 \cdot A_2) \equiv A_1 \times (\text{curl } A_2) + (A_1 \cdot \text{grad})A_2 + A_2 \times (\text{curl } A_1) + (A_2 \cdot \text{grad})A_1$$

## Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho \cos \varphi \quad y = \rho \sin \varphi \quad z = z$	$x = r \cos \varphi \sin \theta \quad y = r \sin \varphi \sin \theta$ $z = r \cos \theta$
Vector $A$	$A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$	$A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{z}$	$A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$
Gradient $\nabla \phi$	$\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$	$\frac{\partial \phi}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} \hat{\varphi} + \frac{\partial \phi}{\partial z} \hat{z}$	$\frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{\varphi}$
Divergence $\nabla \cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial A_\theta \sin \theta}{\partial \theta}$ $+ \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$
Curl $\nabla \times A$	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho} \hat{\rho} & \hat{\varphi} & \frac{1}{\rho} \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2 \sin \theta} \hat{r} & \frac{1}{r \sin \theta} \hat{\theta} & \frac{1}{r} \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & r A_\theta & r A_\varphi \sin \theta \end{vmatrix}$
Laplacian $\nabla^2 \phi$	$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right)$ $+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$

## Transformation of integrals

$L$  = the distance along some curve 'C' in space and is measured from some fixed point.

$S$  = a surface area

$\tau$  = a volume contained by a specified surface

$\hat{\mathbf{t}}$  = the unit tangent to C at the point P

$\hat{\mathbf{n}}$  = the unit outward pointing normal

$A$  = some vector function

$dL$  = the vector element of curve ( $= \hat{\mathbf{t}} dL$ )

$dS$  = the vector element of surface ( $= \hat{\mathbf{n}} dS$ )

Then  $\int_C \mathbf{A} \cdot \hat{\mathbf{t}} dL = \int_C \mathbf{A} \cdot dL$

and when  $A = \nabla \phi$

$$\int_C (\nabla \phi) \cdot dL = \int_C d\phi$$

*Gauss's Theorem (Divergence Theorem)*

When  $S$  defines a closed region having a volume  $\tau$

$$\int_\tau (\nabla \cdot \mathbf{A}) d\tau = \int_S (\mathbf{A} \cdot \hat{\mathbf{n}}) dS = \int_S \mathbf{A} \cdot dS$$

also

$$\int_\tau (\nabla \phi) d\tau = \int_S \phi dS$$

$$\int_\tau (\nabla \times \mathbf{A}) d\tau = \int_S (\hat{\mathbf{n}} \times \mathbf{A}) dS$$

Stokes's Theorem

When  $C$  is closed and bounds the open surface  $S$ ,

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{L}$$

also

$$\int_S (\hat{n} \times \nabla \phi) \cdot d\mathbf{S} = \int_C \phi \cdot d\mathbf{L}$$

Green's Theorem

$$\begin{aligned} \int_S \psi \nabla \phi \cdot d\mathbf{S} &= \int_{\tau} \nabla \cdot (\psi \nabla \phi) \, d\tau \\ &= \int_{\tau} [\psi \nabla^2 \phi + (\nabla \psi) \cdot (\nabla \phi)] \, d\tau \end{aligned}$$

Green's Second Theorem

$$\int_{\tau} (\psi \nabla^2 \phi - \phi \nabla^2 \psi) \, d\tau = \int_S [\psi(\nabla \phi) - \phi(\nabla \psi)] \cdot d\mathbf{S}$$

## Complex Variables

### Complex numbers

The complex number  $z = x + iy = r(\cos \theta + i \sin \theta) = r e^{i(\theta + 2n\pi)}$ , where  $i^2 = -1$  and  $n$  is an arbitrary integer. The real quantity  $r$  is the modulus of  $z$  and the angle  $\theta$  is the argument of  $z$ . The complex conjugate of  $z$  is  $z^* = x - iy = r(\cos \theta - i \sin \theta) = r e^{-i\theta}$ ;  $zz^* = |z|^2 = x^2 + y^2$

### De Moivre's theorem

$$(\cos \theta + i \sin \theta)^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

### Power series for complex variables.

$e^z$	$= 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$	convergent for all finite $z$
$\sin z$	$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$	convergent for all finite $z$
$\cos z$	$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$	convergent for all finite $z$
$\ln(1 + z)$	$= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$	principal value of $\ln(1 + z)$

This last series converges both on and within the circle  $|z| = 1$  except at the point  $z = -1$ .

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots$$

This last series converges both on and within the circle  $|z| = 1$  except at the points  $z = \pm i$ .

$$(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \dots$$

This last series converges both on and within the circle  $|z| = 1$  except at the point  $z = -1$ .



## Trigonometric Formulae

$$\begin{aligned} \cos^2 A + \sin^2 A &= 1 & \sec^2 A - \tan^2 A &= 1 & \operatorname{cosec}^2 A - \cot^2 A &= 1 \\ \sin 2A &= 2 \sin A \cos A & \cos 2A &= \cos^2 A - \sin^2 A & \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A} \end{aligned}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \qquad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos^3 A = \frac{3 \cos A + \cos 3A}{4}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \qquad \sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

### Relations between sides and angles of any plane triangle

In a plane triangle with angles  $A, B,$  and  $C$  and sides opposite  $a, b,$  and  $c$  respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a = b \cos C + c \cos B$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\text{area} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{1}{2}(a+b+c)$$

### Relations between sides and angles of any spherical triangle

In a spherical triangle with angles  $A, B,$  and  $C$  and sides opposite  $a, b,$  and  $c$  respectively,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

## Hyperbolic Functions

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

valid for all  $x$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

valid for all  $x$

$$\cosh ix = \cos x$$

$$\cos ix = \cosh x$$

$$\sinh ix = i \sin x$$

$$\sin ix = i \sinh x$$

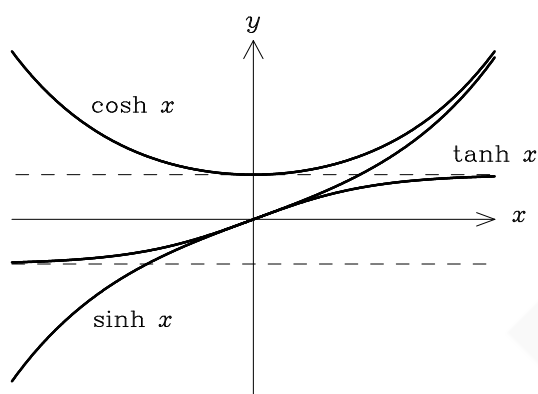
$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$



For large positive  $x$ :

$$\cosh x \approx \sinh x \rightarrow \frac{e^x}{2}$$

$$\tanh x \rightarrow 1$$

For large negative  $x$ :

$$\cosh x \approx -\sinh x \rightarrow \frac{e^{-x}}{2}$$

$$\tanh x \rightarrow -1$$

### Relations of the functions

$$\sinh x = -\sinh(-x)$$

$$\operatorname{sech} x = \operatorname{sech}(-x)$$

$$\cosh x = \cosh(-x)$$

$$\operatorname{cosech} x = -\operatorname{cosech}(-x)$$

$$\tanh x = -\tanh(-x)$$

$$\operatorname{coth} x = -\operatorname{coth}(-x)$$

$$\sinh x = \frac{2 \tanh(x/2)}{1 - \tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}}$$

$$\cosh x = \frac{1 + \tanh^2(x/2)}{1 - \tanh^2(x/2)} = \frac{1}{\sqrt{1 - \tanh^2 x}}$$

$$\tanh x = \sqrt{1 - \operatorname{sech}^2 x}$$

$$\operatorname{sech} x = \sqrt{1 - \tanh^2 x}$$

$$\operatorname{coth} x = \sqrt{\operatorname{cosech}^2 x + 1}$$

$$\operatorname{cosech} x = \sqrt{\operatorname{coth}^2 x - 1}$$

$$\sinh(x/2) = \sqrt{\frac{\cosh x - 1}{2}}$$

$$\cosh(x/2) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh(2x) = 2 \sinh x \cosh x$$

$$\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$\sinh(3x) = 3 \sinh x + 4 \sinh^3 x$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

$$\tanh(3x) = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y) \quad \cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$$

$$\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y) \quad \cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$$

$$\sinh x \pm \cosh x = \frac{1 \pm \tanh(x/2)}{1 \mp \tanh(x/2)} = e^{\pm x}$$

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$

$$\coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y}$$

### Inverse functions

$$\sinh^{-1} \frac{x}{a} = \ln \left( \frac{x + \sqrt{x^2 + a^2}}{a} \right) \quad \text{for } -\infty < x < \infty$$

$$\cosh^{-1} \frac{x}{a} = \ln \left( \frac{x + \sqrt{x^2 - a^2}}{a} \right) \quad \text{for } x \geq a$$

$$\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln \left( \frac{a + x}{a - x} \right) \quad \text{for } x^2 < a^2$$

$$\coth^{-1} \frac{x}{a} = \frac{1}{2} \ln \left( \frac{x + a}{x - a} \right) \quad \text{for } x^2 > a^2$$

$$\operatorname{sech}^{-1} \frac{x}{a} = \ln \left( \frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1} \right) \quad \text{for } 0 < x \leq a$$

$$\operatorname{cosech}^{-1} \frac{x}{a} = \ln \left( \frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1} \right) \quad \text{for } x \neq 0$$

### Limits

$$n^c x^n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ if } |x| < 1 \text{ (any fixed } c)$$

$$x^n/n! \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (any fixed } x)$$

$$(1 + x/n)^n \rightarrow e^x \text{ as } n \rightarrow \infty, x \ln x \rightarrow 0 \text{ as } x \rightarrow 0$$

$$\text{If } f(a) = g(a) = 0 \text{ then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \text{ (l'H\^opital's rule)}$$

## Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \dots + {}^nC_r u^{(n-r)}v^{(r)} + \dots + uv^{(n)}$$

Leibniz Theorem

$$\text{where } {}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

## Integration

### Standard forms

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

for  $n \neq -1$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int \ln x dx = x(\ln x - 1) + c$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

$$\int x e^{ax} dx = e^{ax} \left( \frac{x}{a} - \frac{1}{a^2} \right) + c$$

$$\int x \ln x dx = \frac{x^2}{2} \left( \ln x - \frac{1}{2} \right) + c$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right) + c = \frac{1}{2a} \ln \left( \frac{a+x}{a-x} \right) + c$$

for  $x^2 < a^2$

$$\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \coth^{-1} \left( \frac{x}{a} \right) + c = \frac{1}{2a} \ln \left( \frac{x-a}{x+a} \right) + c$$

for  $x^2 > a^2$

$$\int \frac{x}{(x^2 \pm a^2)^n} dx = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c$$

for  $n \neq 1$

$$\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left( x + \sqrt{x^2 \pm a^2} \right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left[ x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \left( \frac{x}{a} \right) \right] + c$$

$$\int_0^{\infty} \frac{1}{(1+x)x^p} dx = \pi \operatorname{cosec} p\pi$$

for  $p < 1$

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) dx = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2) dx = \begin{cases} 1 \times 3 \times 5 \times \dots \times (n-1) \sigma^{n+1} \sqrt{2\pi} \\ 0 \end{cases}$$

for  $n \geq 2$  and even

for  $n \geq 1$  and odd

$$\int \sin x dx = -\cos x + c$$

$$\int \sinh x dx = \cosh x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \cosh x dx = \sinh x + c$$

$$\int \tan x dx = -\ln(\cos x) + c$$

$$\int \tanh x dx = \ln(\cosh x) + c$$

$$\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cot x) + c$$

$$\int \operatorname{cosech} x dx = \ln[\tanh(x/2)] + c$$

$$\int \sec x dx = \ln(\sec x + \tan x) + c$$

$$\int \operatorname{sech} x dx = 2 \tan^{-1}(e^x) + c$$

$$\int \cot x dx = \ln(\sin x) + c$$

$$\int \operatorname{coth} x dx = \ln(\sinh x) + c$$

$$\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c$$

if  $m^2 \neq n^2$

$$\int \cos mx \cos nx dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c$$

if  $m^2 \neq n^2$

### Standard substitutions

If the integrand is a function of: substitute:

$$(a^2 - x^2) \text{ or } \sqrt{a^2 - x^2} \quad x = a \sin \theta \text{ or } x = a \cos \theta$$

$$(x^2 + a^2) \text{ or } \sqrt{x^2 + a^2} \quad x = a \tan \theta \text{ or } x = a \sinh \theta$$

$$(x^2 - a^2) \text{ or } \sqrt{x^2 - a^2} \quad x = a \sec \theta \text{ or } x = a \cosh \theta$$

If the integrand is a rational function of  $\sin x$  or  $\cos x$  or both, substitute  $t = \tan(x/2)$  and use the results:

$$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad dx = \frac{2 dt}{1+t^2}$$

If the integrand is of the form: substitute:

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \quad px+q = u^2$$

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \quad ax+b = \frac{1}{u}$$

## Integration by parts

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

## Differentiation of an integral

If  $f(x, \alpha)$  is a function of  $x$  containing a parameter  $\alpha$  and the limits of integration  $a$  and  $b$  are functions of  $\alpha$  then

$$\frac{d}{d\alpha} \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) \, dx = f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) \, dx.$$

Special case,

$$\frac{d}{dx} \int_a^x f(y) \, dy = f(x).$$

## Dirac $\delta$ -'function'

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \tau)] \, d\omega.$$

If  $f(t)$  is an arbitrary function of  $t$  then  $\int_{-\infty}^{\infty} \delta(t - \tau) f(t) \, dt = f(\tau)$ .

$\delta(t) = 0$  if  $t \neq 0$ , also  $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$

## Reduction formulae

### Factorials

$$n! = n(n-1)(n-2) \dots 1, \quad 0! = 1.$$

Stirling's formula for large  $n$ :  $\ln(n!) \approx n \ln n - n$ .

For any  $p > -1$ ,  $\int_0^{\infty} x^p e^{-x} \, dx = p \int_0^{\infty} x^{p-1} e^{-x} \, dx = p!$ .  $(-1/2)! = \sqrt{\pi}$ ,  $(1/2)! = \sqrt{\pi}/2$ , etc.

For any  $p, q > -1$ ,  $\int_0^1 x^p (1-x)^q \, dx = \frac{p!q!}{(p+q+1)!}$ .

### Trigonometrical

If  $m, n$  are integers,

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} \theta \cos^n \theta \, d\theta = \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m \theta \cos^{n-2} \theta \, d\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{1}{2}, \quad \int_0^{\pi/2} \sin \theta \, d\theta = 1, \quad \int_0^{\pi/2} \cos \theta \, d\theta = 1, \quad \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

### Other

If  $I_n = \int_0^{\infty} x^n \exp(-\alpha x^2) \, dx$  then  $I_n = \frac{(n-1)}{2\alpha} I_{n-2}$ ,  $I_0 = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$ ,  $I_1 = \frac{1}{2\alpha}$ .

## Differential Equations

### Diffusion (conduction) equation

$$\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi$$

### Wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

### Legendre's equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + l(l + 1)y = 0,$$

solutions of which are Legendre polynomials  $P_l(x)$ , where  $P_l(x) = \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l$ , Rodrigues' formula so  $P_0(x) = 1$ ,  $P_1(x) = x$ ,  $P_2(x) = \frac{1}{2}(3x^2 - 1)$  etc.

Recursion relation

$$P_l(x) = \frac{1}{l} [(2l - 1)xP_{l-1}(x) - (l - 1)P_{l-2}(x)]$$

Orthogonality

$$\int_{-1}^1 P_l(x) P_l(x) dx = \frac{2}{2l + 1} \delta_{ll'}$$

### Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0,$$

solutions of which are Bessel functions  $J_m(x)$  of order  $m$ .

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \quad (\text{integer } m).$$

The same general form holds for non-integer  $m > 0$ .

## Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = [A\rho^n + B\rho^{-n}][C \exp(in\varphi) + D \exp(-in\varphi)]$$

where  $A, B, C, D$  are constants and  $n$  is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = [Ar^l + Br^{-(l+1)}]P_l^m [C \sin m\varphi + D \cos m\varphi]$$

where  $l$  and  $m$  are integers with  $l \geq |m| \geq 0$ ;  $A, B, C, D$  are constants;

$$P_l^m(\cos \theta) = \sin^{|m|} \theta \left[ \frac{d}{d(\cos \theta)} \right]^{|m|} P_l(\cos \theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)[A \cos m\varphi + B \sin m\varphi][C \exp(nz) + D \exp(-nz)]$$

where  $m$  and  $n$  are integers;  $A, B, C, D$  are constants.

## Spherical harmonics

The normalized solutions  $Y_l^m(\theta, \varphi)$  of the equation

$$\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_l^m + l(l+1)Y_l^m = 0$$

are called spherical harmonics, and have values given by

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^m(\cos \theta) e^{im\varphi} \times \begin{cases} (-1)^m & \text{for } m \geq 0 \\ 1 & \text{for } m < 0 \end{cases}$$

i.e.,  $Y_0^0 = \sqrt{\frac{1}{4\pi}}$ ,  $Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$ ,  $Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\varphi}$ , etc.

*Orthogonality*

$$\int_{4\pi} Y_l^{*m} Y_{l'}^{m'} d\Omega = \delta_{ll'} \delta_{mm'}$$

## Calculus of Variations

The condition for  $I = \int_a^b F(y, y', x) dx$  to have a stationary value is  $\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)$ , where  $y' = \frac{dy}{dx}$ . This is the Euler-Lagrange equation.



## Functions of Several Variables

If  $\phi = f(x, y, z, \dots)$  then  $\frac{\partial \phi}{\partial x}$  implies differentiation with respect to  $x$  keeping  $y, z, \dots$  constant.

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \dots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x} \delta x + \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial z} \delta z + \dots$$

where  $x, y, z, \dots$  are independent variables.  $\frac{\partial \phi}{\partial x}$  is also written as  $\left(\frac{\partial \phi}{\partial x}\right)_{y, \dots}$  or  $\frac{\partial \phi}{\partial x} \Big|_{y, \dots}$  when the variables kept constant need to be stated explicitly.

If  $\phi$  is a well-behaved function then  $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$  etc.

If  $\phi = f(x, y)$ ,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial \phi}\right)_y}, \quad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

### Taylor series for two variables

If  $\phi(x, y)$  is well-behaved in the vicinity of  $x = a, y = b$  then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + \frac{1}{2!} \left( u^2 \frac{\partial^2 \phi}{\partial x^2} + 2uv \frac{\partial^2 \phi}{\partial x \partial y} + v^2 \frac{\partial^2 \phi}{\partial y^2} \right) + \dots$$

where  $x = a + u, y = b + v$  and the differential coefficients are evaluated at  $x = a, y = b$

### Stationary points

A function  $\phi = f(x, y)$  has a stationary point when  $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$ . Unless  $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0$ , the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left. \begin{array}{l} \text{Minimum: } \frac{\partial^2 \phi}{\partial x^2} > 0, \text{ or } \frac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum: } \frac{\partial^2 \phi}{\partial x^2} < 0, \text{ or } \frac{\partial^2 \phi}{\partial y^2} < 0, \end{array} \right\} \text{ and } \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2$$

$$\text{Saddle point: } \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2$$

If  $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0$  the character of the turning point is determined by the next higher derivative.

### Changing variables: the chain rule

If  $\phi = f(x, y, \dots)$  and the variables  $x, y, \dots$  are functions of independent variables  $u, v, \dots$  then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} + \dots$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial v} + \dots$$

etc.

## Changing variables in surface and volume integrals – Jacobians

If an area  $A$  in the  $x, y$  plane maps into an area  $A'$  in the  $u, v$  plane then

$$\int_A f(x, y) \, dx \, dy = \int_{A'} f(u, v) J \, du \, dv \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian  $J$  is also written as  $\frac{\partial(x, y)}{\partial(u, v)}$ . The corresponding formula for volume integrals is

$$\int_V f(x, y, z) \, dx \, dy \, dz = \int_{V'} f(u, v, w) J \, du \, dv \, dw \quad \text{where now} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

## Fourier Series and Transforms

### Fourier series

If  $y(x)$  is a function defined in the range  $-\pi \leq x \leq \pi$  then

$$y(x) \approx c_0 + \sum_{m=1}^M c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx$$

where the coefficients are

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \, dx$$

$$c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos mx \, dx$$

$$s_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin mx \, dx$$

$$(m = 1, \dots, M)$$

$$(m = 1, \dots, M')$$

with convergence to  $y(x)$  as  $M, M' \rightarrow \infty$  for all points where  $y(x)$  is continuous.

### Fourier series for other ranges

Variable  $t$ , range  $0 \leq t \leq T$ , (i.e., a periodic function of time with period  $T$ , frequency  $\omega = 2\pi/T$ ).

$$y(t) \approx c_0 + \sum c_m \cos m\omega t + \sum s_m \sin m\omega t$$

where

$$c_0 = \frac{\omega}{2\pi} \int_0^T y(t) \, dt, \quad c_m = \frac{\omega}{\pi} \int_0^T y(t) \cos m\omega t \, dt, \quad s_m = \frac{\omega}{\pi} \int_0^T y(t) \sin m\omega t \, dt.$$

Variable  $x$ , range  $0 \leq x \leq L$ ,

$$y(x) \approx c_0 + \sum c_m \cos \frac{2m\pi x}{L} + \sum s_m \sin \frac{2m\pi x}{L}$$

where

$$c_0 = \frac{1}{L} \int_0^L y(x) \, dx, \quad c_m = \frac{2}{L} \int_0^L y(x) \cos \frac{2m\pi x}{L} \, dx, \quad s_m = \frac{2}{L} \int_0^L y(x) \sin \frac{2m\pi x}{L} \, dx.$$

### Fourier series for odd and even functions

If  $y(x)$  is an *odd* (anti-symmetric) function [i.e.,  $y(-x) = -y(x)$ ] defined in the range  $-\pi \leq x \leq \pi$ , then only sines are required in the Fourier series and  $s_m = \frac{2}{\pi} \int_0^\pi y(x) \sin mx \, dx$ . If, in addition,  $y(x)$  is symmetric about  $x = \pi/2$ , then the coefficients  $s_m$  are given by  $s_m = 0$  (for  $m$  even),  $s_m = \frac{4}{\pi} \int_0^{\pi/2} y(x) \sin mx \, dx$  (for  $m$  odd). If  $y(x)$  is an *even* (symmetric) function [i.e.,  $y(-x) = y(x)$ ] defined in the range  $-\pi \leq x \leq \pi$ , then only constant and cosine terms are required in the Fourier series and  $c_0 = \frac{1}{\pi} \int_0^\pi y(x) \, dx$ ,  $c_m = \frac{2}{\pi} \int_0^\pi y(x) \cos mx \, dx$ . If, in addition,  $y(x)$  is anti-symmetric about  $x = \frac{\pi}{2}$ , then  $c_0 = 0$  and the coefficients  $c_m$  are given by  $c_m = 0$  (for  $m$  even),  $c_m = \frac{4}{\pi} \int_0^{\pi/2} y(x) \cos mx \, dx$  (for  $m$  odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

### Complex form of Fourier series

If  $y(x)$  is a function defined in the range  $-\pi \leq x \leq \pi$  then

$$y(x) \approx \sum_{-M}^M C_m e^{imx}, \quad C_m = \frac{1}{2\pi} \int_{-\pi}^\pi y(x) e^{-imx} \, dx$$

with  $m$  taking all integer values in the range  $\pm M$ . This approximation converges to  $y(x)$  as  $M \rightarrow \infty$  under the same conditions as the real form.

For other ranges the formulae are:

Variable  $t$ , range  $0 \leq t \leq T$ , frequency  $\omega = 2\pi/T$ ,

$$y(t) = \sum_{-\infty}^{\infty} C_m e^{im\omega t}, \quad C_m = \frac{\omega}{2\pi} \int_0^T y(t) e^{-im\omega t} \, dt.$$

Variable  $x'$ , range  $0 \leq x' \leq L$ ,

$$y(x') = \sum_{-\infty}^{\infty} C_m e^{i2m\pi x'/L}, \quad C_m = \frac{1}{L} \int_0^L y(x') e^{-i2m\pi x'/L} \, dx'.$$

### Discrete Fourier series

If  $y(x)$  is a function defined in the range  $-\pi \leq x \leq \pi$  which is sampled in the  $2N$  equally spaced points  $x_n = nx/N$  [ $n = -(N-1) \dots N$ ], then

$$y(x_n) = c_0 + c_1 \cos x_n + c_2 \cos 2x_n + \dots + c_{N-1} \cos(N-1)x_n + c_N \cos Nx_n \\ + s_1 \sin x_n + s_2 \sin 2x_n + \dots + s_{N-1} \sin(N-1)x_n + s_N \sin Nx_n$$

where the coefficients are

$$c_0 = \frac{1}{2N} \sum y(x_n)$$

$$c_m = \frac{1}{N} \sum y(x_n) \cos mx_n \quad (m = 1, \dots, N-1)$$

$$c_N = \frac{1}{2N} \sum y(x_n) \cos Nx_n$$

$$s_m = \frac{1}{N} \sum y(x_n) \sin mx_n \quad (m = 1, \dots, N-1)$$

$$s_N = \frac{1}{2N} \sum y(x_n) \sin Nx_n$$

each summation being over the  $2N$  sampling points  $x_n$ .

### Fourier transforms

If  $y(x)$  is a function defined in the range  $-\infty \leq x \leq \infty$  then the Fourier transform  $\hat{y}(\omega)$  is defined by the equations

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t} d\omega, \quad \hat{y}(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} dt.$$

If  $\omega$  is replaced by  $2\pi f$ , where  $f$  is the frequency, this relationship becomes

$$y(t) = \int_{-\infty}^{\infty} \hat{y}(f) e^{i2\pi ft} df, \quad \hat{y}(f) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi ft} dt.$$

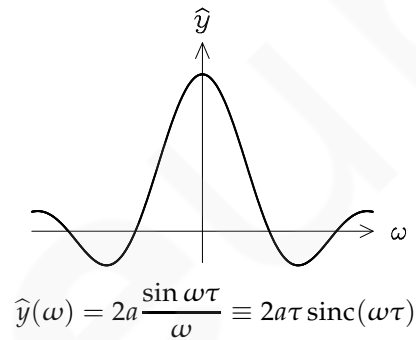
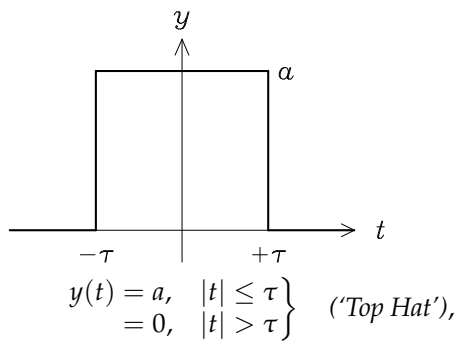
If  $y(t)$  is symmetric about  $t = 0$  then

$$y(t) = \frac{1}{\pi} \int_0^{\infty} \hat{y}(\omega) \cos \omega t d\omega, \quad \hat{y}(\omega) = 2 \int_0^{\infty} y(t) \cos \omega t dt.$$

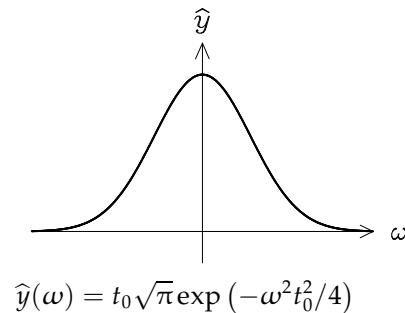
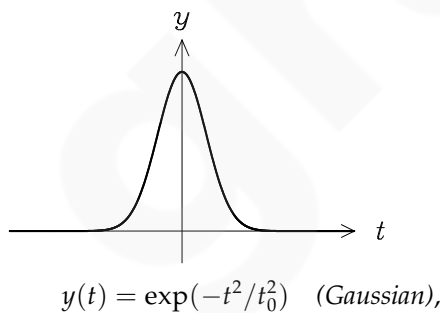
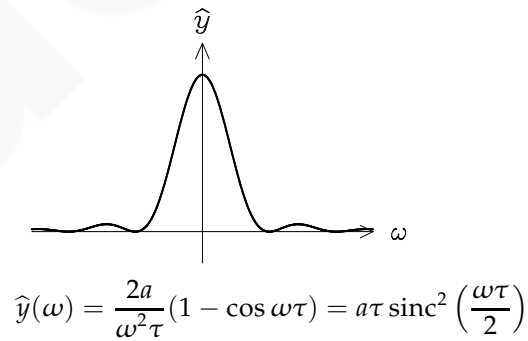
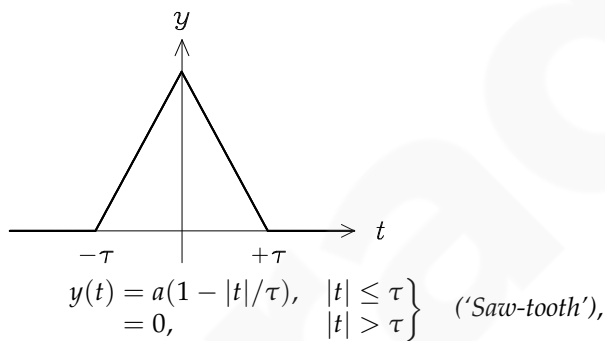
If  $y(t)$  is anti-symmetric about  $t = 0$  then

$$y(t) = \frac{1}{\pi} \int_0^{\infty} \hat{y}(\omega) \sin \omega t d\omega, \quad \hat{y}(\omega) = 2 \int_0^{\infty} y(t) \sin \omega t dt.$$

Specific cases



where  $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$



$y(t) = f(t) e^{i\omega_0 t}$  (modulated function),

$\hat{y}(\omega) = \hat{f}(\omega - \omega_0)$

$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau)$  (sampling function)

$\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)$

**Convolution theorem**

If  $z(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau) d\tau = \int_{-\infty}^{\infty} x(t - \tau)y(\tau) d\tau \equiv x(t) * y(t)$  then  $\widehat{z}(\omega) = \widehat{x}(\omega) \widehat{y}(\omega)$ .

Conversely,  $\widehat{x\widehat{y}} = \widehat{x} * \widehat{y}$ .

**Parseval's theorem**

$$\int_{-\infty}^{\infty} y^*(t) y(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{y}^*(\omega) \widehat{y}(\omega) d\omega$$

(if  $\widehat{y}$  is normalised as on page 21)

**Fourier transforms in two dimensions**

$$\begin{aligned} \widehat{V}(\mathbf{k}) &= \int V(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^2\mathbf{r} \\ &= \int_0^{\infty} 2\pi r V(r) J_0(kr) dr \quad \text{if azimuthally symmetric} \end{aligned}$$

**Fourier transforms in three dimensions**

$$\begin{aligned} \widehat{V}(\mathbf{k}) &= \int V(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r} \\ &= \frac{4\pi}{k} \int_0^{\infty} V(r) r \sin kr dr \quad \text{if spherically symmetric} \\ V(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int \widehat{V}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k} \end{aligned}$$

**Examples**

$V(\mathbf{r})$	$\widehat{V}(\mathbf{k})$
$\frac{1}{4\pi r}$	$\frac{1}{k^2}$
$\frac{e^{-\lambda r}}{4\pi r}$	$\frac{1}{k^2 + \lambda^2}$
$\nabla V(\mathbf{r})$	$i\mathbf{k}\widehat{V}(\mathbf{k})$
$\nabla^2 V(\mathbf{r})$	$-k^2\widehat{V}(\mathbf{k})$

## Laplace Transforms

If  $y(t)$  is a function defined for  $t \geq 0$ , the Laplace transform  $\bar{y}(s)$  is defined by the equation

$$\bar{y}(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt$$

Function $y(t)$ ( $t > 0$ )	Transform $\bar{y}(s)$	
$\delta(t)$	1	Delta function
$\theta(t)$	$\frac{1}{s}$	Unit step function
$t^n$	$\frac{n!}{s^{n+1}}$	
$t^{1/2}$	$\frac{1}{2} \sqrt{\frac{\pi}{s^3}}$	
$t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$	
$e^{-at}$	$\frac{1}{(s+a)}$	
$\sin \omega t$	$\frac{\omega}{(s^2 + \omega^2)}$	
$\cos \omega t$	$\frac{s}{(s^2 + \omega^2)}$	
$\sinh \omega t$	$\frac{\omega}{(s^2 - \omega^2)}$	
$\cosh \omega t$	$\frac{s}{(s^2 - \omega^2)}$	
$e^{-at} y(t)$	$\bar{y}(s+a)$	
$y(t-\tau) \theta(t-\tau)$	$e^{-s\tau} \bar{y}(s)$	
$t y(t)$	$-\frac{d\bar{y}}{ds}$	
$\frac{dy}{dt}$	$s\bar{y}(s) - y(0)$	
$\frac{d^n y}{dt^n}$	$s^n \bar{y}(s) - s^{n-1} y(0) - s^{n-2} \left[ \frac{dy}{dt} \right]_0 \dots - \left[ \frac{d^{n-1} y}{dt^{n-1}} \right]_0$	
$\int_0^t y(\tau) d\tau$	$\frac{\bar{y}(s)}{s}$	
$\left. \begin{aligned} &\int_0^t x(\tau) y(t-\tau) d\tau \\ &\int_0^t x(t-\tau) y(\tau) d\tau \end{aligned} \right\}$	$\bar{x}(s) \bar{y}(s)$	Convolution theorem

[Note that if  $y(t) = 0$  for  $t < 0$  then the Fourier transform of  $y(t)$  is  $\hat{y}(\omega) = \bar{y}(i\omega)$ .]

## Numerical Analysis

### Finding the zeros of equations

If the equation is  $y = f(x)$  and  $x_n$  is an approximation to the root then either

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{(Newton)}$$

$$\text{or, } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \quad \text{(Linear interpolation)}$$

are, in general, better approximations.

### Numerical integration of differential equations

If  $\frac{dy}{dx} = f(x, y)$  then

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{where } h = x_{n+1} - x_n \quad \text{(Euler method)}$$

$$\text{Putting } y_{n+1}^* = y_n + hf(x_n, y_n) \quad \text{(improved Euler method)}$$

$$\text{then } y_{n+1} = y_n + \frac{h[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)]}{2}$$

### Central difference notation

If  $y(x)$  is tabulated at equal intervals of  $x$ , where  $h$  is the interval, then  $\delta y_{n+1/2} = y_{n+1} - y_n$  and  $\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2}$

### Approximating to derivatives

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h} \approx \frac{y_n - y_{n-1}}{h} \approx \frac{\delta y_{n+1/2} + \delta y_{n-1/2}}{2h} \quad \text{where } h = x_{n+1} - x_n$$

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = \frac{\delta^2 y_n}{h^2}$$

### Interpolation: Everett's formula

$$y(x) = y(x_0 + \theta h) \approx \bar{\theta}y_0 + \theta y_1 + \frac{1}{3!}\bar{\theta}(\bar{\theta}^2 - 1)\delta^2 y_0 + \frac{1}{3!}\theta(\theta^2 - 1)\delta^2 y_1 + \dots$$

where  $\theta$  is the fraction of the interval  $h (= x_{n+1} - x_n)$  between the sampling points and  $\bar{\theta} = 1 - \theta$ . The first two terms represent linear interpolation.

### Numerical evaluation of definite integrals

#### Trapezoidal rule

The interval of integration is divided into  $n$  equal sub-intervals, each of width  $h$ ; then

$$\int_a^b f(x) dx \approx h \left[ c \frac{1}{2} f(a) + f(x_1) + \dots + f(x_j) + \dots + \frac{1}{2} f(b) \right]$$

$$\text{where } h = (b - a)/n \text{ and } x_j = a + jh.$$

#### Simpson's rule

The interval of integration is divided into an even number (say  $2n$ ) of equal sub-intervals, each of width  $h = (b - a)/2n$ ; then

$$\int_a^b f(x) dx \approx \frac{h}{3} [f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)]$$

Gauss's integration formulae

These have the general form  $\int_{-1}^1 y(x) dx \approx \sum_1^n c_i y(x_i)$

For  $n = 2$ :  $x_i = \pm 0.5773$ ;  $c_i = 1, 1$  (exact for any cubic).

For  $n = 3$ :  $x_i = -0.7746, 0.0, 0.7746$ ;  $c_i = 0.555, 0.888, 0.555$  (exact for any quintic).

### Treatment of Random Errors

Sample mean  $\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$

Residual:  $d = x - \bar{x}$

Standard deviation of sample:  $s = \frac{1}{\sqrt{n}}(d_1^2 + d_2^2 + \dots + d_n^2)^{1/2}$

Standard deviation of distribution:  $\sigma \approx \frac{1}{\sqrt{n-1}}(d_1^2 + d_2^2 + \dots + d_n^2)^{1/2}$

Standard deviation of mean:  $\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n(n-1)}}(d_1^2 + d_2^2 + \dots + d_n^2)^{1/2}$   
 $= \frac{1}{\sqrt{n(n-1)}} \left[ \sum x_i^2 - \frac{1}{n} (\sum x_i)^2 \right]^{1/2}$

Result of  $n$  measurements is quoted as  $\bar{x} \pm \sigma_m$ .

### Range method

A quick but crude method of estimating  $\sigma$  is to find the range  $r$  of a set of  $n$  readings, i.e., the difference between the largest and smallest values, then

$$\sigma \approx \frac{r}{\sqrt{n}}$$

This is usually adequate for  $n$  less than about 12.

### Combination of errors

If  $Z = Z(A, B, \dots)$  (with  $A, B$ , etc. independent) then

$$(\sigma_Z)^2 = \left( \frac{\partial Z}{\partial A} \sigma_A \right)^2 + \left( \frac{\partial Z}{\partial B} \sigma_B \right)^2 + \dots$$

So if

(i)  $Z = A \pm B \pm C$ ,  $(\sigma_Z)^2 = (\sigma_A)^2 + (\sigma_B)^2 + (\sigma_C)^2$

(ii)  $Z = AB$  or  $A/B$ ,  $\left( \frac{\sigma_Z}{Z} \right)^2 = \left( \frac{\sigma_A}{A} \right)^2 + \left( \frac{\sigma_B}{B} \right)^2$

(iii)  $Z = A^m$ ,  $\frac{\sigma_Z}{Z} = m \frac{\sigma_A}{A}$

(iv)  $Z = \ln A$ ,  $\sigma_Z = \frac{\sigma_A}{A}$

(v)  $Z = \exp A$ ,  $\frac{\sigma_Z}{Z} = \sigma_A$



## Statistics

### Mean and Variance

A random variable  $X$  has a distribution over some subset  $x$  of the real numbers. When the distribution of  $X$  is discrete, the probability that  $X = x_i$  is  $P_i$ . When the distribution is continuous, the probability that  $X$  lies in an interval  $\delta x$  is  $f(x)\delta x$ , where  $f(x)$  is the probability density function.

$$\text{Mean } \mu = E(X) = \sum P_i x_i \text{ or } \int x f(x) dx.$$

$$\text{Variance } \sigma^2 = V(X) = E[(X - \mu)^2] = \sum P_i (x_i - \mu)^2 \text{ or } \int (x - \mu)^2 f(x) dx.$$

### Probability distributions

Error function:  $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$

Binomial:  $f(x) = \binom{n}{x} p^x q^{n-x}$  where  $q = (1 - p)$ ,  $\mu = np$ ,  $\sigma^2 = npq$ ,  $p < 1$ .

Poisson:  $f(x) = \frac{\mu^x}{x!} e^{-\mu}$ , and  $\sigma^2 = \mu$

Normal:  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \mu)^2}{2\sigma^2}\right]$

### Weighted sums of random variables

If  $W = aX + bY$  then  $E(W) = aE(X) + bE(Y)$ . If  $X$  and  $Y$  are independent then  $V(W) = a^2V(X) + b^2V(Y)$ .

### Statistics of a data sample $x_1, \dots, x_n$

$$\text{Sample mean } \bar{x} = \frac{1}{n} \sum x_i$$

$$\text{Sample variance } s^2 = \frac{1}{n} \sum (x_i - \bar{x})^2 = \left(\frac{1}{n} \sum x_i^2\right) - \bar{x}^2 = E(x^2) - [E(x)]^2$$

### Regression (least squares fitting)

To fit a straight line by least squares to  $n$  pairs of points  $(x_i, y_i)$ , model the observations by  $y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i$ , where the  $\epsilon_i$  are independent samples of a random variable with zero mean and variance  $\sigma^2$ .

$$\text{Sample statistics: } s_x^2 = \frac{1}{n} \sum (x_i - \bar{x})^2, \quad s_y^2 = \frac{1}{n} \sum (y_i - \bar{y})^2, \quad s_{xy}^2 = \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y}).$$

$$\text{Estimators: } \hat{\alpha} = \bar{y}, \quad \hat{\beta} = \frac{s_{xy}^2}{s_x^2}; \quad E(Y \text{ at } x) = \hat{\alpha} + \hat{\beta}(x - \bar{x}); \quad \hat{\sigma}^2 = \frac{n}{n-2} (\text{residual variance}),$$

$$\text{where residual variance} = \frac{1}{n} \sum \{y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})\}^2 = s_y^2 - \frac{s_{xy}^4}{s_x^2}.$$

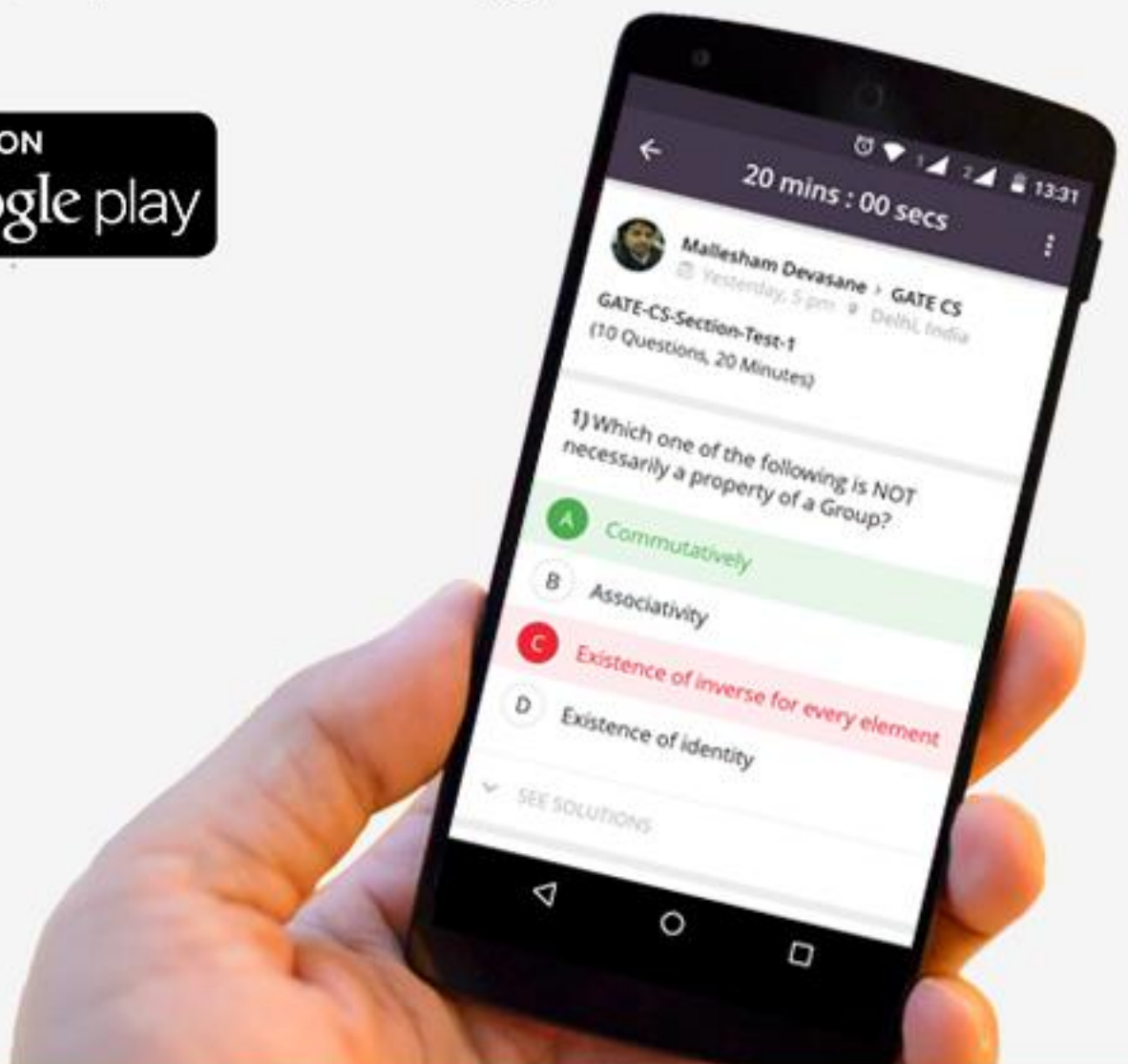
$$\text{Estimates for the variances of } \hat{\alpha} \text{ and } \hat{\beta} \text{ are } \frac{\hat{\sigma}^2}{n} \text{ and } \frac{\hat{\sigma}^2}{ns_x^2}.$$

$$\text{Correlation coefficient: } \hat{\rho} = r = \frac{s_{xy}^2}{s_x s_y}.$$



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