Electromagnetic Fields & Theory

Electromagnetic waves:

Maxwell's equations predict the existence of electro-magnetic waves that propagate in vacuum with the elec-tric and magnetic fields perpendicular and with ratio:

$$E = cB$$

The waves travel with velocity c where

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Energy in Electromagnetic waves:

The energy flow rate (power per unit area) of an elec-tromagnetic wave is given by the Poynting vector $\vec{S}$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

The magnitude of the time-averaged value of $\vec{S}$ is called the intensity of the wave

$$I = \frac{1}{2} \frac{E_{\max} B_{\max}}{\mu_0} = \frac{E_{\max}^2}{2\mu_0 c} = \frac{1}{2} \epsilon_0 c E_{\max}^2$$

Speed of light in materials

When light propagates through a material, its speed is lower than the speed in free space space by a factor called the index of refraction

$$v = \frac{c}{n}$$

Reflection and refraction

At a smooth interface, the incident, reflected, and re-fracted rays and the normal to the interface all lie in a single plane. The angle of incidence and angle of reflec-tion (measured from the normal) are equal $\theta_r = \theta_a$ and the angle of refraction is given by Snell's law:

$$n_a \sin \theta_a = n_b \sin \theta_b$$

Polarization

A polarizing filter passes waves that are linearly po-larized along its polarizing axis. When polarized light of intensity $I_{\max}$ is incident on a polarizing filter used as an analyzer, the intensity I of the light transmitted depends on the angle ϕ between the polarization direction of the incident light and the polarizing axis of the analyzer:

$$I = I_{\max} \cos^2 \phi$$

Spherical Mirrors

Object and image distances:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

where $f = R/2$.

Thin Lenses

Object and image distances:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

where

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Magnification

The lateral magnification for the systems described above is

$$m = \frac{y'}{y} = -\frac{s'}{s}$$

Forces:

The force on a charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is

$$\vec{F} = q\vec{v} \times \vec{B}$$

and the force on a differential segment $d\vec{l}$ carrying current I is

$$d\vec{F} = I d\vec{l} \times \vec{B}$$

Magnetic Flux:

Magnetic flux is defined analogously to electric flux

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

The magnetic flux through a closed surface seems to be zero

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Magnetic dipoles:

A current loop creates a magnetic dipole $\vec{\mu} = I\vec{A}$ where I is the current in the loop and $\vec{A}$ is a vector normal to the plane of the loop and equal to the area of the loop. The torque on a magnetic dipole in a magnetic field is

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Biot-Savart Law:

The magnetic field $d\vec{B}$ produced at point P by a differential segment $d\vec{l}$ carrying current I is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2}$$

where $\hat{r}$ points from the segment $d\vec{l}$ to the point P.

Magnetic field produced by a moving charge:

Similarly, the magnetic field produced at a point P by a moving charge is

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\, \vec{v} \times \hat{r}}{r^2}$$

Ampère's Law: (without displacement current)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}$$

Faraday's Law:

The EMF produced in a closed loop depends on the change of the magnetic flux through the loop

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

When an EMF is produced by a changing magnetic flux there is an induced, nonconservative, electric field $\vec{E}$ such that

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{A}$$

Mutual Inductance:

When a changing current i_1 in circuit 1 causes a changing magnetic flux in circuit 2, and vice-versa, the induced EMF in the circuits is

$$\mathcal{E}_2 = -M\frac{di_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M\frac{di_2}{dt}$$

where M is the *mutual inductance* of the two loops

$$M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_1 \Phi_{B1}}{i_1}$$

where N_i is the number of loops in circuit i.

Self Inductance:

A changing current i in any circuit generates a changing magnetic field that induces an EMF in the circuit:

$$\mathcal{E} = -L\frac{di}{dt}$$

where L is the *self inductance* of the circuit

$$L = N\frac{\Phi_B}{i}$$

For example, for a solenoid of N turns, length l, area A, Ampère's law gives $B = \mu_0(N/l)i$, so the flux is $\Phi_B = \mu_0(N/l)iA$, and so

$$L = \mu_0 \frac{N^2}{l} A$$

Capacitance:

A capacitor is any pair of conductors separated by an insulating material. When the conductors have equal and opposite charges Q and the potential difference between the two conductors is V_{ab}, then the definition of the capacitance of the two conductors is

$$C = \frac{Q}{V_{ab}}$$

The energy stored in the electric field is

$$U = \frac{1}{2}CV^2$$

If the capacitor is made from parallel plates of area A separated by a distance d, where the size of the plates is much greater than d, then the capacitance is given by

$$C = \epsilon_0 A/d$$

Capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + ...$$

Capacitors in parallel:

$$C_{\text{eq}} = C_1 + C_2 + ...$$

If a dielectric material is inserted, then the capacitance increases by a factor of K where K is the dielectric constant of the material

$$C = KC_0$$

Current:

When current flows in a conductor, we define the current as the rate at which charge passes:

$$I = \frac{dQ}{dt}$$

We define the current density as the current per unit area, and can relate it to the drift velocity of charge carriers by

$$\vec{J} = nq\vec{v}_d$$

where n is the number density of charges and q is the charge of one charge carrier.

Ohm's Law and Resistance:

Ohm's Law states that a current density J in a material is proportional to the electric field E. The ratio $\rho = E/J$ is called the *resistivity* of the material. For a conductor with cylindrical cross section, with area A and length L, the *resistance* R of the conductor is

$$R = \frac{\rho L}{A}$$

A current I flowing through the resistor R produces a potential difference V given by

$$V = IR$$

Resistors in series:

$$R_{\text{eq}} = R_1 + R_2 + ...$$

Resistors in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + ...$$

Power:

The power transferred to a component in a circuit by a current I is

$$P = VI$$

where V is the potential difference across the component.

Kirchhoff's rules:

The algebraic sum of the currents into any junction must be zero:

$$\sum I = 0$$

The algebraic sum of the potential differences around any loop must be zero.

$$\sum V = 0$$

Force on a charge:

An electric field $\vec{E}$ exerts a force $\vec{F}$ on a charge q given by:

$$\vec{F} = q\vec{E}$$

Coulomb's law:

A point charge q located at the coordinate origin gives rise to an electric field $\vec{E}$ given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}$$

where r is the distance from the origin (spherical coordinate), $\hat{r}$ is the spherical unit vector, and ϵ_0 is the permittivity of free space:

$$\epsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C}^2/(\mathrm{N}\cdot\mathrm{m}^2)$$

Superposition:

The principle of superposition of electric fields states that the electric field $\vec{E}$ of any combination of charges is the vector sum of the fields caused by the individual charges

$$\vec{E} = \sum_i \vec{E}_i$$

To calculate the electric field caused by a continuous distribution of charge, divide the distribution into small elements and integrate all these elements:

$$\vec{E} = \int d\vec{E} = \int_q \frac{dq}{4\pi\epsilon_0 r^2}\,\hat{r}$$

Electric flux:

Electric flux is a measure of the "flow" of electric field through a surface. It is equal to the product of the area element and the perpendicular component of $\vec{E}$ integrated over a surface:

$$\Phi_E = \int E\cos\phi\, dA = \int \vec{E}\cdot\hat{n}\, dA = \int \vec{E}\cdot d\vec{A}$$

where ϕ is the angle from the electric field $\vec{E}$ to the surface normal $\hat{n}$.

Gauss' Law:

Gauss' law states that the total electric flux through any closed surface is determined by the charge enclosed by that surface:

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

Electric conductors:

The electric field inside a conductor is zero. All excess charge on a conductor resides on the surface of that conductor.

Electric Potential:

The electric potential is defined as the potential energy per unit charge. If the electric potential at some point is V then the electric potential energy at that point is $U = qV$. The electric potential function $V(\vec{r})$ is given by the line integral:

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l} + V(\vec{r}_0)$$

Beware of the minus sign. This gives the potential produced by a point charge q:

$$V = \frac{q}{4\pi\epsilon_0 r}$$

for a collection of charges q_i

$$V = \sum_i \frac{q_i}{4\pi\epsilon_0 r_i}$$

and for a continuous distribution of charge

$$V = \int_q \frac{dq}{4\pi\epsilon_0 r}$$

where in each of these cases, the potential is taken to be zero infinitely far from the charges.

Field from potential:

If the electric potential function is known, the vector electric field can be derived from it:

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \quad E_z = -\frac{\partial V}{\partial z}$$

or in vector form:

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right)$$

Beware of the minus sign.

Coulomb's Law: The electric field intensity of a point charge is in the outward radial direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge.

Gauss's Law: The net electric flux through any closed surface is equal to 1/ε times the net electric charge enclosed within that closed surface.

Maxwell Equations

Differential form	integral form	Significance
$\nabla \times E = -\frac{\partial B}{\partial t}$	$\oint E \cdot dl = -\frac{d\Phi}{dt}$	Faraday's Law
$\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint H \cdot dl = I + \oint_s \frac{\partial D}{\partial t} \cdot ds$	Ampere's Circuital Law
$\nabla \cdot D = \rho_v$	$\oint D \cdot ds = Q$	Gaus's Law
$\nabla . B = 0$	$\oint B \cdot ds = 0$	No isolated magnetic charge

Electrostatics

Force on a point charge q inside a static electric field

$$\mathbf{F} = q\mathbf{E}$$

Gauss's law

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \quad \text{or} \quad \nabla \cdot \mathbf{D} = \rho$$

Electrostatic fields are conservative

$$\nabla \times \mathbf{E} = 0 \quad \text{or} \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

Electric field produced by a point charge q in free space

$$\mathbf{E} = \frac{q\left(\mathbf{R} - \mathbf{R_i}\right)}{4\pi\epsilon_0 \left|\mathbf{R} - \mathbf{R_i}\right|^3}$$

Electric field produced by a volume charge distribution

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \int_{V'} \hat{\mathbf{R}}' \frac{\rho_v \; dV'}{R'^2}$$

Electric field produced by a surface charge distribution

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \; ds'}{R'^2}$$

Electric field produced by a line charge distribution

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_l \; dl'}{R'^2}$$

Electric field produced by an infinite sheet of charge

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon}$$

Electric field produced by an infinite line of charge

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \hat{\mathbf{r}} \frac{D_r}{\epsilon} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\epsilon r}$$

Electric field - scalar potential relationship

$$\mathbf{E} = -\nabla V \quad \text{or} \quad V_2 - V_1 = -\int_{P1}^{P2} \mathbf{E} \cdot d\mathbf{l}$$

Electric potential due to a point charge (with infinity chosen as the reference)

$$V = \frac{q}{4\pi\epsilon_0 \left|\mathbf{R} - \mathbf{R_i}\right|}$$

Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Constitutive relationship in dielectric materials

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

where $\mathbf{P}$ is the polarization.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

Electrostatic energy density

$$w_e = \frac{1}{2} \epsilon E^2$$

Boundary conditions

$$E_{1t} = E_{2t} \quad \text{or} \quad \hat{\mathbf{n}} \times (\mathbf{E_1} - \mathbf{E_2}) = 0$$

$$D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad \hat{\mathbf{n}} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$$

Ohm's law

$$\mathbf{J} = \sigma \mathbf{E}$$

Conductivity

$$\sigma = \rho_v \mu$$

where μ stands for charge mobility.

Joule's law

$$P = \int \mathbf{E} \cdot \mathbf{J} \; dv$$

Electrostatics

$$\mathbf{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\mathbf{a}_{R_{12}} \quad , \mathbf{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r}-\mathbf{r}_k)}{|\mathbf{r}-\mathbf{r}_k|^3} \; ,$$

$$\mathbf{E} = \frac{\mathbf{F}}{Q}, \; \mathbf{E} = \int\frac{\rho_L dl}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \; , \mathbf{E} = \int\frac{\rho_S dS}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \; , \mathbf{E} = \int\frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \quad \mathbf{E} = \frac{\rho_S}{2\varepsilon_0}\mathbf{a}_n, \; \mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho$$

$$Q = \oint_S \mathbf{D}\cdot d\mathbf{S} = \int_v \rho_v dv, \; \nabla\cdot\mathbf{D} = \rho_v, \; W = -Q\int_A^B \mathbf{E}\cdot d\ell,$$

$$W = -Q\int_A^B \mathbf{E}\cdot d\ell, \quad V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E}\cdot d\ell, \; V = \frac{Q}{4\pi\varepsilon_0 r} \quad \oint \mathbf{E}\cdot d\ell = 0, \; \nabla\times\mathbf{E} = 0, \; \mathbf{E} = -\nabla V$$

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k \quad , W_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E} dv = \frac{1}{2}\int \varepsilon_0 E^2 dv$$

$$\mathbf{J} = \rho_v\mathbf{u}, \; I = \int_S \mathbf{J}\cdot d\mathbf{S}, \; \mathbf{J} = \sigma\mathbf{E},$$

$$R = \frac{\mathrm{V}}{\mathrm{I}} = \frac{\int \mathbf{E}\cdot d\mathbf{l}}{\int \sigma\mathbf{E}\cdot d\mathbf{S}}$$

$$\mathbf{D} = \varepsilon\,\mathbf{E}, \quad \nabla\cdot\mathbf{J} = -\frac{\partial\rho_v}{\partial t}, \; E_{1t} = E_{2t}, \; D_{1n} - D_{2n} = \rho_S, \; D_{1n} = D_{2n}, \; \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}, \; \nabla^2 V = 0,$$

$$C = \frac{Q}{V} = \frac{\varepsilon\oint \mathbf{E}\cdot d\mathbf{S}}{\int \mathbf{E}\cdot d\mathbf{l}}, \; W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}, C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}, C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}, RC = \frac{\varepsilon}{\sigma}$$

Boundary Conditions for Electrostatic Fields:

$E_{1t} = E_{2t}$ States that Tangential component of an E field is continuous across an interface. $\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$

$D_{1n} - D_{2n} = \rho_s$ States that the Normal component of D field is discontinuous across an interface where surface charge exist-amount the amount of discontinuity being equal to the surface charge density.

When two dielectrics are in contact of no free charges at interface, $\rho_s = 0$. Then $D_{1n} = D_{2n}$ or $\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$

Magnetostatics:

Force on a moving charge q inside a magnetic field

$$\mathbf{F} = q\mathbf{u} \times \mathbf{B}$$

Force on an infinitesimally small current element $Id\mathbf{l}$ inside a magnetic field

$$d\mathbf{F}_m = Id\mathbf{l} \times \mathbf{B}$$

Torque on a N-turn loop carrying current I inside a uniform magnetic field

$$\mathbf{T} = \mathbf{m} \times \mathbf{B}$$

where $\mathbf{m} = \hat{\mathbf{n}}NIA$.

Gauss's law for magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \text{or} \quad \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \text{or} \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = I$$

Magnetic flux density — magnetic vector potential relationship

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Magnetic potential produced by a current distribution

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}}{R'} dV'$$

Vector Poisson's Equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Magnetic field intensity produced by an infinitesimally small current element (Biot-Savart law)

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2}$$

Magnetic field produced by an infinitely long wire of current in the z-direction

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r}$$

Magnetic field produced by a circular loop of current in the ϕ-direction

$$\mathbf{H} = \hat{\mathbf{z}} \frac{Ia^2}{2(a^2 + z^2)^{3/2}}$$

Constitutive relationship in magnetic materials

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

Magnetization

$$\mathbf{M} = \chi_m \mathbf{H}$$

Boundary conditions

$$B_{1n} = B_{2n} \quad \text{or} \quad \hat{\mathbf{n}} \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$$

$$H_{1t} - H_{2t} = J_s \quad \text{or} \quad \hat{\mathbf{n}} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s}$$

Magnetostatic energy density

$$w_m = \frac{1}{2} \mu H^2$$

Magnetostatics:

$$\mathbf{H}=\int_L \frac{Id\mathbf{l}\times\mathbf{a}_R}{4\pi R^2},\ \mathbf{H}=\int_S \frac{\mathbf{K}dS\times\mathbf{a}_R}{4\pi R^2},\ \mathbf{H}=\int_v \frac{\mathbf{J}dv\times\mathbf{a}_R}{4\pi R^2}$$

$$\mathbf{H}=\frac{I}{4\pi\rho}(\cos\alpha_2-\cos\alpha_1)\mathbf{a}_\phi,\ \mathbf{H}=\frac{I}{2\pi\rho}\mathbf{a}_\phi,\ \mathbf{a}_\phi=\mathbf{a}_\ell\times\mathbf{a}_\rho,$$

$$\oint \mathbf{H}\cdot d\mathbf{l}=I_{enc},\ \nabla\times\mathbf{H}=\mathbf{J},\ \mathbf{H}=\frac{I}{2\pi\rho}\mathbf{a}_\phi,\ \mathbf{H}=\frac{1}{2}\mathbf{K}\times\mathbf{a}_n$$

$$\mathbf{B}=\mu\mathbf{H},\ \Psi=\int_S \mathbf{B}\cdot d\mathbf{S},\ \oint \mathbf{B}\cdot d\mathbf{S}=0,\ \nabla\cdot\mathbf{B}=0,\ \mathbf{H}=-\nabla V_m,$$

$$\mathbf{B}=\nabla\times\mathbf{A},\ \mathbf{A}=\int_L \frac{\mu_0 Id\mathbf{l}}{4\pi R},\ \mathbf{A}=\int_S \frac{\mu_0\mathbf{K}dS}{4\pi R},\ \mathbf{A}=\int_v \frac{\mu_0\mathbf{J}dv}{4\pi R},\ \Psi=\oint_L \mathbf{A}\cdot d\mathbf{l}$$

$$\mathbf{F}=Q(\mathbf{E}+\mathbf{u}\times\mathbf{B}),\ d\mathbf{F}=Id\mathbf{l}\times\mathbf{B},\qquad \mathbf{B}_{1n}=\mathbf{B}_{2n},$$

$$(\mathbf{H}_1-\mathbf{H}_2)\times\mathbf{a}_{n12}=\mathbf{K},\ \mathbf{H}_{1t}=\mathbf{H}_{2t},\ \frac{\tan\theta_1}{\tan\theta_2}=\frac{\mu_1}{\mu_2}$$

$$L=\frac{\lambda}{I}=\frac{N\psi}{I},\ M_{12}=\frac{\lambda_{12}}{I_2}=\frac{N_1\psi_{12}}{I_2},\ W_m=\frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}dv=\frac{1}{2}\int \mu H^2 dv$$

Boundary Conditions for Magnetostatic Fields:

$B_{1n}=B_{2n}$ States that Normal component of B is continuous across an interface. $\mu_1 H_{1n}=\mu_2 H_{2n}$

$H_{1t}-H_{2t}=J_{sn}$ States that the Tangential component of H field is discontinuous across an interface where free surface current exist-amount the amount of discontinuity being equal to the surface current density.

When conductivities of both media are finite, current are defined by volume current densities and free surface currents don't exist on interface hence j equal to zero, and the Tangential component of H field is continuous across the boundary of almost all physical media; it is discontinuous only when an interface with an ideal conductor or a super conductor is assumed.

Constants

Free space permittivity $\epsilon_0 = 8.85 \times 10^{-12}$ F/m

Free space permeability $\mu_0 = 4\pi \times 10^{-7}$ H/m

Differential operations:

Gradient of a scalar field: The vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar as the gradient of that scalar.
Measures the rate and direction of change in a scalar field. Maps scalar fields to vector fields.

Divergence of the vector field: the divergence of a vector field A at a point, abbreviated div A, as the net outward flux of A per unit volume as the volume about the point tends to zero:
Measures the scalar of a source or sink at a given point in a vector field. Maps vector fields to scalar fields.

Curl of a vector field: The curl of u vector field A, denoted by Curl A , is a vector whose magnitude is the maximum net circulation of A per unit urea as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum.
Measures the tendency to rotate about a point in a vector field. Maps vector fields to (pseudo)vector fields.

Divergence theorem: The volume integral of divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

Stoke theorem: The surface integral of the curl of vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

Coordinate Systems	Gradient	divergence
Rectangular	$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$	$\nabla . A = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$
Rectangular	$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{\partial V}{r\partial \emptyset} a_\emptyset + \frac{\partial V}{\partial z} a_z$	$\nabla . A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{r\partial \emptyset} A_\emptyset + \frac{\partial}{\partial z} A_z$
Spherical	$\nabla V = \frac{\partial V}{\partial R} a_R + \frac{\partial V}{R\partial \theta} a_\theta + \frac{\partial V}{R sin\theta \partial \emptyset} a_\emptyset$	$\nabla V = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R sin\theta}\frac{\partial}{R\partial \theta}(A_\theta sin\theta) + \frac{\partial}{R sin\theta \partial \emptyset} A_\emptyset$

Gradient, Divergence, Curl and Laplacian Operators

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Electromagnetic wave propagation

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$\times$ H = J + $\dot{D}$
$\times$ E = - $\dot{B}$
∇. D = ρ_v
∇.B = 0

D = εE
B = μH
J = σE

^{2}E = μεE
^{2}H = μεH

$\frac{E_y}{H_z} = -\frac{E_z}{H_y} = \sqrt{\mu/\varepsilon}$; E.H = 0 E ⊥ H in UPW

For loss less medium $^2E - \rho^2 E = 0$ $\rho = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$.

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right)}$$

$E(z, t) = E_0\, e^{-\alpha z}\cos(\omega t - \beta z)$; $H_0 = E_0 / \eta$.

$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$ $|\eta| < \theta_\eta$

$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}}$ $\tan 2\theta_\eta = \sigma/\omega\varepsilon$.

η= α + jβ α → attenuation constant → Neper /m . $|N_p| = 20\log_{10} e = 8.686$ dB

For loss less medium σ = 0; α = 0.

β → phase shift/length ; μ = ω / β ; λ = 2π/β .

$\frac{J_s}{J_d} = \left|\frac{\sigma E}{j\omega\epsilon E}\right| = \sigma / \omega\epsilon = \tan\theta$ → loss tanjent $\theta = 2\theta_\eta$

If tan θ is very small (σ < < ωϵ) → good (lossless) dielectric

If tan θ is very large (σ >> ωϵ) → good conductor

Complex permittivity $\epsilon_C = \epsilon\left(1 - \frac{j\sigma}{\omega\epsilon}\right) = \varepsilon' - j\,\varepsilon''$.

Tan $\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\epsilon}$.

Plane wave in loss less dielectric :- (σ ≈ 0)

$\alpha = 0$; $\beta = \omega\sqrt{\mu\epsilon}$; $\omega = \frac{}{\sqrt{\mu\epsilon}}$; $\lambda = 2\pi/\beta$; $\eta = \sqrt{\mu_r/\varepsilon_r}\ \angle 0$.

E & H are in phase in lossless dielectric

Free space :- ($\sigma = 0$, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$)

$\alpha = 0$, $\beta = \omega\sqrt{\mu_0\varepsilon_0}$; $u = 1/\sqrt{\mu_0\varepsilon_0}$, $\lambda = 2\pi/\beta$; $\eta = \sqrt{\mu_0/\varepsilon_0} < 0 = 120\pi\angle 0$

Here also E & H in phase .

Good Conductor :-

$\sigma >> \omega\epsilon$ $\sigma/\omega\epsilon \to \infty \Rightarrow \sigma = \infty$ $\varepsilon = \varepsilon_0$; $\mu = \mu_0\mu_r$

$\alpha = \beta = \sqrt{\pi f\mu\sigma}$; $u = \sqrt{2\omega/\mu\sigma}$; $\lambda = 2\pi/\beta$; $\eta = \sqrt{\frac{W\mu}{\sigma}}\angle 45^0$

Skin depth $\delta = 1/\alpha$

$\eta = \frac{}{\sigma\delta}\sqrt{2}\, e^{j\pi/4} = \frac{+j}{\sigma\delta}$

Skin resistance $R_s = \frac{}{\sigma\delta} = \sqrt{\frac{\pi f\mu}{\sigma}}$

$R_{ac} = \frac{R_s.l}{w}$

$R_{dc} = \frac{l}{\sigma s}$.

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Poynting Vector :-

$$\oint_S (E \times H)\ ds = -\frac{\partial}{dt}\int_v \frac{1}{2}[\,\varepsilon E^2 + \mu H^2]\ dv - \int \sigma E^2\ dv$$

$$\delta_{ave}(z) = \frac{1}{2}\frac{E_0^2}{|\eta|}\, e^{-2\alpha z} \cos\theta_\eta\ a_z$$

Total time avge power crossing given area $P_{avge} = \int_S P_{ave}(s)\ ds$

Direction of propagation :- (a_k)

$a_k \times a_E = a_H$

$a_E \times a_H = a_k$

→ Both E & H are normal to direction of propagation

→ Means they form EM wave that has no E or H component along direction of propagation .

Reflection of plane wave :-

(a) Normal incidence

Reflection coefficient $\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

T_{xn} coefficient $T = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$

Medium-I Dielectric , Medium-2 Conductor :-

$\eta_2 > \eta_1$:-

$\Gamma > 0$, there is a standing wave in medium & T_{xed} wave in medium '2'.

Max values of $|E_1|$ occurs

$Z_{max} = -n\pi/\beta_1 = \frac{-n\lambda_1}{2}$; n = 0, 1, 2....

$Z_{min} = \frac{-(2n+1)\pi}{2\beta_1} = \frac{-(2n+1)\lambda_1}{4}$

$\eta_2 < \eta_1$:- E $_{max}$ occurs @ $\beta\ Z_{max} = \frac{-(2n+1)\pi}{2} \Rightarrow Z_{max} = \frac{-(2n+1)\pi}{2\beta_1} = \frac{-(2n+1)\lambda_1}{4}$

$\beta_1 Z_{min} = n\pi \Rightarrow Z_{min} = \frac{-n\pi}{\beta_1} = \frac{-n\lambda_1}{2}$

H min occurs when there is $|t_1|$max

$S = \frac{|E_1|_{max}}{|E_1|_{min}} = \frac{|H_1|_{max}}{|H_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$; $|\Gamma| = \frac{s-1}{s+1}$

Since $|\Gamma| < 1 \Rightarrow 1 \le \delta \le \infty$

Transmission Lines

Supports only TEM mode

$LC = \mu\varepsilon$; $G/C = \sigma/\varepsilon$.

$\frac{d^2V_s}{dz^2} - r^2 V_s = 0$; $\frac{d^2I_s}{dz^2} - r^2 I_s = 0$

$\Gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$

$V(z, t) = V_0^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_0^- e^{\alpha z} \cos(\omega t + \beta z)$

$Z_0 = -\frac{V_0^-}{I_0^-} = \frac{R+j\omega L}{\gamma} = \frac{\gamma}{G+j\omega C} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$

Lossless Line : (R = 0 =G; σ = 0)

→ $\gamma = \alpha + j\beta = j\omega\sqrt{LC}$; $\alpha = 0, \beta = w\sqrt{LC}$; $\lambda = 1/f\sqrt{LC}$, $u = 1/\sqrt{LC}$

$Z_0 = \sqrt{L/C}$

Distortion less :(R/L = G/C)

→ $\alpha = \sqrt{RG}$; $\beta = \omega L\sqrt{\frac{G}{R}} = \omega C\sqrt{\frac{R}{G}} = \omega\sqrt{LC}$

→ $Z_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$; $\lambda = 1/f\sqrt{LC}$; $u = \frac{1}{\sqrt{LC}} = V_p$; $uz_0 = 1/C$, $u/z_0 = 1/L$

i/p impedance :-

$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tan hl}{Z_0 + Z_L \tan hl}\right]$ for lossless line $\gamma = j\beta \Rightarrow \tan hj\beta l = j \tan \beta l$

$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + Z_L \tan \beta l}\right]$

VSWR $= \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$

CSWR = $-\Gamma_L$

Transmission coefficient $S = 1 + \Gamma$

SWR = $\frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}} = \frac{+|\Gamma_L|}{-|\Gamma_L|} = \frac{Z_L}{Z_0} = \frac{Z_0}{Z_L}$

$(Z_L > Z_0)$ $(Z_L < Z_0)$

$|Z_{in}|_{max} = \frac{V_{max}}{I_{min}} = SZ_0$

$|Z_{in}|_{min} = \frac{V_{min}}{I_{max}} = Z_0/S$

Shorted line :- $\Gamma_L = -1$, $S = \infty$ $Z_{in} = Z_{sc} = jZ_0 \tan \beta l$

$\Gamma_L = -1$, $S = \infty$ $Z_{in} = Z_{sc} = j\, Z_0 \tan \beta l$.

Z_{in} may be inductive or capacitive based on length '0'

If $l < \lambda / 4 \rightarrow$ inductive (Z_{in} +ve)

$\frac{\lambda}{4} < l < \lambda/2 \rightarrow$ capacitive (Z_{in} -ve)

Open circuited line :-

$Z_{in} = Z_{oc} = -jZ_0 \cot \beta l$

$\Gamma_l = 1$ $s = \infty$ $l < \lambda / 4$ capacitive

$\frac{\lambda}{4} < l < \lambda/2$ inductive

$Z_{sc}\, Z_{oc} = Z_0^2$

Matched line : $(Z_L = Z_0)$

$Z_{in} = Z_0$ $\Gamma = 0$; $s = 1$

No reflection . Total wave T_{xed} . So, max power transfer possible .

Behaviour of Transmission Line for Different lengths :-

$l = \lambda /4 \rightarrow \left.\begin{matrix} Z_{sc} = \infty \\ Z_{oc} = 0 \end{matrix}\right\} \rightarrow$ impedance inverter @ $l = \lambda /4$

$l = \lambda /2 :$ $Z_{in} = Z_0 \Rightarrow \left.\begin{matrix} Z_{sc} = 0 \\ Z_{oc} = \infty \end{matrix}\right\}$ impedance reflector @ $l = \lambda /2$

Wave Guides

TM modes : ($H_z = 0$)

$$E_z = E_0 \sin\left(\frac{m\pi}{a}\right)x \ \sin\left(\frac{n\pi}{b}\right) y\, e^{-nz}$$

$$h^2 = k_x^2 + k_y^2 \qquad \therefore \gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon} \quad \text{where } k = \omega\sqrt{\mu\epsilon}$$

m→ no. of half cycle variation in X-direction

n→ no. of half cycle variation in Y- direction .

Cut off frequency $\omega_C = \frac{}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ $\quad \gamma = 0; \ \alpha = 0 = \beta$

$k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \rightarrow$ Evanscent mode ; $\gamma = \alpha$; $\beta = 0$

$k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \rightarrow$ Propegation mode $\gamma = j\beta \quad \alpha = 0$

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$f_c = \frac{u_p'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ $\quad u_p'$ = phase velocity = $\frac{}{\sqrt{\mu\epsilon}}$ is lossless dielectric medium

$$\lambda_c = u'/f_c = \frac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}}$$

$\beta = \beta'\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$ $\quad \beta' = \omega/\ W \qquad \beta'$ = phase constant in dielectric medium.

$u_p = \omega/\beta \quad \lambda = 2\pi/\beta = u_p/f \rightarrow$ phase velocity & wave length in side wave guide

$$\eta_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$\eta_{TM} = \eta'\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$ $\quad \eta' \rightarrow$ impedance of UPW in medium

TE Modes :- ($E_z = 0$)

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-nz}$$

$\rightarrow \eta_{TE} = \frac{w\mu}{\beta} = \eta' / \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

$\rightarrow \eta_{TE} > \eta_{TM}$

TE_{10} Dominant mode

Antennas :

Hertzian Dipole :- $H_{\Phi s} = \frac{jI_0\beta\, dl}{4\pi r}\sin\theta\, e^{-j\beta\gamma}$ $\qquad E_{\theta s} = \eta H_{\Phi s}$

Half wave Dipole :-

$H_{\phi s} = \frac{jI_0 e^{-j\beta\gamma}\cos\left(\frac{\pi}{2}\cos\theta\right)}{2\pi\gamma\sin\theta}$; $\qquad E_{\theta s} = \eta H_{\Phi s}$